Math 331.2 Spring 2010: Review problems

Exercise 1 Consider the harmonic oscillator

$$4\frac{d^2y}{dt^2} + k\frac{dy}{dt} + 5y = 0$$

where k is a parameter with $0 \le k < \infty$. As k varies describe the different types of the systems (damped, overdamped, undamped) and the bifurcations.

Exercise 2 Consider the harmonic oscillator

$$\frac{d^2y}{dt^2} + k\frac{dy}{dt} + 2ky = 0$$

where k is a parameter with $0 \le k < \infty$. As k varies describe the different types of the systems (damped, overdamped, undamped) and the bifurcations.

Exercise 3 Compute the general solution for the

$$\begin{aligned}
(a) \quad & \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = e^{-4t} \\
(b) \quad & \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = e^{2t} \\
(c) \quad & \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = e^{t} + e^{-2t} \\
(d) \quad & \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = t^2 + 1
\end{aligned}$$

Exercise 4 Consider the forced harmonic oscillator

$$\frac{dy^2}{dt^2} + 5y = 6\sin(\alpha t)\,.$$

where α is a parameter.

1. For which value of α does the system exhibit a *resonnance*?
2. Find the general solution for the value of α found in (a)

Exercise 5 Consider the forced harmonic osciilator

$$\frac{dy^2}{dt^2} + 4\frac{dy}{dt} + 7y = 6\sin(3t)\,.$$

1. Find the general solution.
2. Describe the behavior of the general solution as $t \to \infty$ and graph a typical solution.
3. Compute the amplitude and phase angle of the *steady state solution*.

Exercise 6 Solve the initial value problem

$$\frac{dy^2}{dt^2} - 4\frac{dy}{dt} - 5y \;=\; 6\sin(3t)\,,\;\; y(0) = 2, y'(0) = -1\,.$$

Exercise 7 Consider the equation

$$\frac{dy^2}{dt^2} + 8y \;=\; 6\sin(3t)\,.$$

1. Determine the frequency of the *beating*.
2. Determine the frequency of the *rapid oscillations*.
3. Give a rough sketch of typical solution indicating clearly the results obtained in 1. and 2.

Remark: To answer this questions you do not need to compute the solutions explicitly.

Exercise 8 Find the general solution of

$$(a)\;\frac{dy^2}{dt^2} + 16y \;=\; 3\sin(4t)\,.$$

$$(b)\;\frac{dy^2}{dt^2} + 16y \;=\; 5\cos(2t)$$

Exercise 9 Solve the initial value problem

$$(a)\;\frac{dy^2}{dt^2} + 16y \;=\; 3\sin(4t)\,,\;\; y(0) = 1, y'(0) = 0$$

$$(b)\;\frac{dy^2}{dt^2} + 16y \;=\; 5\cos(2t),\;\; y(0) = 2, y'(0) = 2$$

Exercise 10 Consider the linear systems $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and A is given by

$$(a)\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad (b)\begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} \quad (c)\begin{pmatrix} 1 & 1/4 \\ -17 & 2 \end{pmatrix} \quad (d)\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \quad (e)\begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}$$

In each case of the five cases

1. Determine the type of the system, i.e., sink, source, saddle, center, spiral source, spiral sink, center, degenerate eigenvalues.
2. Draw the phase portrait of the system. If the eigenvalue are real you need to compute the eigenvectors and indicate them clearly on the phase portrait.
3. Draw a rough graph of a typical solution $x(t)$, $y(t)$. Note that you *do not* need to solve the system to do this! If the eigenvalues are complex indicate clearly in your graph the period of the oscillations.

Exercise 11 Consider the linear systems $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$ where A is given by

$$(a)\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad (b)\begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} \quad (c)\begin{pmatrix} 1 & 1/4 \\ -17 & 2 \end{pmatrix} \quad (c)\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

In each case

1. Compute the *general solution* of $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$.

2. Solve the *initial value problem* $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Exercise 12 Consider the linear systems $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$ where A is given by

$$(a)\begin{pmatrix} 1 & -\frac{9}{4} \\ \alpha & -4 \end{pmatrix} \quad (b)\begin{pmatrix} a & 3a \\ 1 & 2 \end{pmatrix}$$

and α is a parameter. Compute the eigenvalues as a function of α to d etermine the various types of the systems and the bifurcations as the parameter α varies.

Exercise 13 Compute the inverse Laplace transform of the following functions

$$(a)\ \frac{7}{s+2} \quad (b)\ \frac{e^{-8s}}{s(s+2)} \quad (c)\ \frac{e^{-5s}}{s^2+2s-2} \quad (d)\ \frac{1}{s^2+2s+2} \quad (e)\ \frac{e^{-2s}}{s^2+2s+2}$$

$$(g)\ \frac{2s-5}{s^2+2s+2} \quad (h)\ \frac{e^{-3s}}{(s^2+1)(s^2+4)} \quad (i)\frac{e^{-2s}}{(s-1)(s^2+4s+5)}$$

Exercise 14 The function $h(t)$ is given by

$$h(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 1 \\ 2 & \text{if} \quad 1 \le t < 3 \\ 1 & \text{if} \quad 3 \le t\,. \end{cases}$$

1. Compute the Laplace transform of $h(t)$. *Hint:* Write h as a combination of $u_a(t)$ for suitable a's.

2. Solve the equation $\dfrac{dy}{dt} + 3y = h(t)$.

Exercise 15 Use the Laplace transform method to solve the following initial value problems.

1. $\dfrac{dy}{dt} + 5y = 5u_2(t)\,,\ y(0) = -7\,.$ Make also a graph of the solutions.

2. $\dfrac{dy}{dt} + 4y = -3u_4(t)e^{2(t-4)}\,,\ y(0) = 2\,.$ What is $\lim_{t\to\infty} y(t)$?

3. $\dfrac{dy^2}{dt^2} + 4y = 2u_2(t)\cos(3(t-2)) \quad y(0) = 0,\ y'(0) = 1\,.$

4. $\dfrac{dy^2}{dt^2} + 4y \;=\; 3u_1(t)e^{-(t-1)} \quad y(0) = 0,\; y'(0) = 1\,.$ How does the solution behave for large t?

5. $\dfrac{dy^2}{dt^2} + 2\dfrac{dy}{dt} + 10y \;=\; u_4(t) \quad y(0) = 2,\; y'(0) = 0\,.$ What is $\lim_{t\to\infty} y(t)$? Make a graph of the solution.

6. $\dfrac{dy^2}{dt^2} + 5y \;=\; \delta(t-3) \quad y(0) = 2,\; y'(0) = 1\,.$ Make a graph of the solution.

7. $\dfrac{dy^2}{dt^2} + 4\dfrac{dy}{dt} + 7y \;=\; \delta(t-5) \quad y(0) = 6,\; y'(0) = -1\,.$ Make a graph of the solution.