MATH 331.2, Fall 2008: Practice problems

These problems are selected problems from quizzes and midterms in the last few years.

1. Solve the initial value problem $\dfrac{dy}{dt} = -3y - \cos(2t)\,,\ y(0) = 4.$

2. Solve the initial value problem $\dfrac{dy}{dt} = -e^t/y\,,\ y(0) = -2.$

3. Consider the differential equation $\dfrac{dy}{dt} = y^2 - 4y + 2\,.$
(a) Find all the equilibrium points, determine their type (sink, source, node).
(b) Draw the phase line of the system.
(c) Sketch the solutions of the equation with the initial conditions (a) y(0)= 1, (b) y(0)=-5, and (c) y(0)=6. Indicate clearly the behavior for $t \to \infty$ and $t \to -\infty$.

4. A home buyer can afford to spend no more than \$1000 per month on mortgage payments. Suppose that the interest rate is 5% (per year) and that the term of the mortgage is 20 years. Assume that interest is compounded continuously and that payments are also made continuously.
(a) Determine the maximum amount that this buyer can afford to borrow.
(b) Determine the total interest paid during the term of the mortgage

5. Solve the initial value problem $\dfrac{dx}{dt} = x^2\,[t + \sin(t)]$ with $x(0) = 6$.

6. Mary initially deposits \$1000 in a savings account that pays interest at the rate of 5% per year (compounded continuously). She also arranges for \$25 per week to be deposited automatically into her account.
(a) Assume that weekly deposits can be approximated by continuous deposits. Write down an initial value problem for her account balance $S(t)$ over time (t measured in years).
(b) How long does she needs to save to buy a \$5000 car?

7. Consider the following equation for a certain population of squirrels given by $P(t)$ (t is measured in years).

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{2}\right)(P - 1)$$

(a) Find all the equilibrium points of the equations. Draw the phase line and indicate the type of each equilibrium points (i.e., sink, source, or node).
(b) Make a graph of the solutions with initial conditions $P(0) = 1/4$, $P(0) = 3/2$, and $P(0) = 3$.

(c) At a certain time the hunting of squirrels become permitted and the law allows that a certain percentage α of the squirrel population be eliminated every year. A new equation for the squirrel population is then

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{2}\right)(P-1) - \alpha P$$

The IALS (International Association for the Liberation of Squirrels) asserts than no more than 10% of squirrels should be eliminated every year (i.e $\alpha = 0.1$), otherwise the population would go extinct. On the contrary the UHA (United Hunters of America) asserts that it is safe to hunt half of the squirrel population every year (i.e. $\alpha = 0.5$).

Analyze the bifurcations of the systems as α varies and determine who is right from the IALS or the UHA.

8. Find the solution for the system

$$\begin{aligned} \frac{dx}{dt} &= xy + y \\ \frac{dy}{dt} &= 2 \end{aligned}$$

with $x(0) = 3$ and $y(0) = 0$.

9. Solve the initial value problem $\dfrac{dy}{dt} = 5y + 12e^{3t}\,, \quad y(0) = -3\,.$

10. Consider the equation $\dfrac{dy}{dt} = y^2 + \alpha y^2$. where $-\infty < \alpha < \alpha$ is a parameter. Identify the bifurcation values of α and describe the bifuractions that take place as α increases. Draw representative phase lines for α before the bifurcation ,at the bifurcation and after the bifurcation.

11. (a) Find the general solution of $\dfrac{dy}{dt} = (2y-1)(1+t)$

(b) Solve the initial value problem $\dfrac{dy}{dt} = (2y-1)(1+t)\,,\ y(0) = 2\,.$

12. Consider the differential equation $\dfrac{dy}{dt} = y^2(y-1)$

(a) Find the equilibrium points, draw the phase line, and identify the equilibrium points as source, sink, or node.

(b) Sketch the solutions with initial conditions $y(0) = -1$, $y(0) = 1/2$, $y(0) = 2$.

13. Solve the initial value problem $\dfrac{dy}{dt} = 6y + e^t + 2e^{2t}\,, \quad y(0) = 3\,.$

14. A couple of years ago, after careful analysis, a team of scientists came up with the following systems of differential equations describing the interaction between populations of frogs and alligators in a swamp. The alligators eat the frogs. The

population of frogs is denoted by $F(t)$ and the population of alligators is denoted by $A(t)$; and t is measured in years.

$$\begin{aligned}\frac{dF}{dt} &= 6F - 10AF \\ \frac{dA}{dt} &= -2A + 1.2AF\end{aligned}$$

(a) The intensive use of pesticides in the past few years has weakened the frogs considerably. The rate at which they reproduce now is only 1/4 what it used to be and when interacting with alligators they are now twice as likely to be killed and eaten as in the past.
Write an equation describing the population of frogs under these new conditions.

$\frac{dF}{dt} =$

(b) Worried about the low numbers of frogs, the team of scientists decide, in addition, to introduce 10 frogs per month from another region.
Write an equation describing the population of frogs under these new conditions.

$\frac{dF}{dt} =$

15. Solve the initial value problem $\frac{dy}{dt} = \frac{1+y^2}{y}\,, \quad y(0) = -2\,.$

16. Consider the differential equation $\frac{dy}{dt} = 3y^3 - 12y$
(a) Find the equilibrium points, draw the phase line, and identify the equilibrium points as source, sink, or node.
(b) Sketch the solutions with initial conditions $y(0) = 2$, $y(0) = -1$, $y(0) = -3$.

17. Solve the general solution of $\frac{dy}{dt} = t^2 + \frac{2}{t}y$

18. Find the solution for the system

$$\begin{aligned}\frac{dx}{dt} &= 5x \\ \frac{dy}{dt} &= 2y + 3x + 1\end{aligned}$$

with $x(0) = 3$ and $y(0) = 1$.