Math 331.2: Homework 4 (Section 2.5)

In each of the following four problems you should (a) Graph $f(y)$ versus y. (b) Determine all the critical points and classify them as asymptotically stable or unstable. (c) Draw the phase line and sketch several graphs of solutions $y(t)$, $y(t)$ versus t. You should sketch enough graph to have a representative for each kind of solutions.

1. $\frac{dy}{dt} = y(y+2)(y-3)$.

2. $\frac{dy}{dt} = e^y - 1$.

3. $\frac{dy}{dt} = -y^2 + \frac{7}{2}y - \frac{3}{2}$.

4. $\frac{dy}{dt} = y(1-y^2)$.

5. Consider the equation

$$\frac{dy}{dt} = 2(1-y)^2 \tag{1}$$

(a) Find the critical points and draw the graph of $f(y) = 2(1-y)^2$. Is the critical point asymptotically stable or asymptotically unstable? Draw the phase line and the graph of some solution $y(t)$.

(b) Solve the equation (1) with initial condition y_0 and discuss the stability of the critical point using your solution.

6. Consider the equation

$$\frac{dy}{dt} = ry\ln(K/y) \tag{2}$$

(a) Find the critical points and draw the graph of $f(y) = ry\ln(K/y)$. Determine if the cricitical points are stable or unstable. Draw the phase line and the graph of some solution $y(t)$.

(b) Determine where the graph of $y(t)$ is concave up or concave down.

(c) Solve the equation (2) with initial value y_0.

7. Solve the initial value problem $\frac{dy}{dt} = -y(1-y), y(0) = y_0$.

8. A population of fish (measured in millions, without harvesting) is described by the logistic equation

$$\frac{dy}{dt} = 3y(1-y/5)\,.$$

We model the harvesting of fishes by assuming that a certain proportion α of fish is harvested every year.

(a) Write down an equation which model the evolution of the fish population with harvesting.

(b) Find the critical points for the model with harvesting and determine their stability. How do the critical points change with α?

(c) Determine which is the maximum level of harvesting that does not lead to population collapse?

9. The velocity of object in free fall near the surface of the earth and subject to air friction satisfies the equation

$$\frac{dv}{dt} = g + f(v) \tag{3}$$

To model friction the function $f(v)$ should satisfy $f(0) = 0$ (no velocity =no friction), $f(v)$ increases with v (the stronger the velocity the stronger the frictio) and finally $f(v) < 0$ if $v > 0$ and $f(v) > 0$ if $v < 0$ which means that the friction force $f(v)$ alaways produces a deceleration of the object. Using this information and the phase line analysis show that the object in free always reaches a limiting velocity and determines an equation for the limiting velocity.

Hints and solutions:
5 (a) The only critical point is 1 which is neither stable nor unstable. If $y_0 < 1$ then $y(t)$ approaches 1 as t grows while if $y_0 > 1$ $y(t)$ diverges away from 1

5 (b) The solution is $y(t) = 1 - \dfrac{1 - y_0}{2t(1 - y_0) + 1}$

6 To solve use $u = \ln(y/K)$ and the solution is $y(t) = Ke^{\ln(y_0/K)e^{-rt}}$.

8 (a) $\dfrac{dy}{dt} = 3y(1 - y/5) - \alpha y$.

8 (c) The maximum sustainable harvesting level is $\alpha = 3$.