Math 331.2: Homework 12 (Section 7.5)

For the following systems

(a) Determine if the systems are *a saddle, a source (also called stable node) or a sink (unstable node).*

(b) Sketch typical solution in in the phase plane (the x_1-x_2 plane). Indicate clearly the eigenvectors and sketch some typical solutions.

(c) Write down the general solutions.

1. $\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$

2. $\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$

3. $\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

4. $\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x}$

For the following systems find the solution of the initial value problem. Sketch the graph of the two components $x_1(t)$ and $x_2(t)$ of the solutions $\mathbf{x}(t)$ and describe their behavior as $t \to \infty$.

5. $\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

6. $\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

7. Consider the second order equation

$$ay'' + by' + cy = 0 . \tag{1}$$

Define the new variable $v = y'$ and set

$$\mathbf{x} = \begin{pmatrix} y \\ v \end{pmatrix}$$

Show that the second order equation (1) is equivalent to the system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \mathbf{x} \tag{2}$$

Show that the characteristic polynomial to determine the eiegenvalue for the system (2) is the equation $a\lambda^2 + b\lambda + c = 0$.

8. Consider the system $\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \mathbf{x}$.

(a) Solve the system for $\alpha = 0.5$. Write down the general solution, classify the type of the system (saddle, source, etc...) and draw the phase portrait.

(b) Solve the system for $\alpha = 2$. Write down the general solution, classify the type of the system (saddle, source, etc...) and draw the phase portrait.

(c) The solutions of the systems in part (a) and (b) are quite different. Find the eigenvalues as a function of α and determine for which value of α the system change from a behavior of the type found in (a) to a behavior of the type found in (b). This critical value of α is called a **bifurcation point**.