Math 331.2: Homework 1 (Section 1.1 and 1.2)

1. Sketch the vector a direction field for the equations
(a) $y' = -1 - 2y$
(b) $y' = 3y + 1$.
Using the direction field to describe the behavior of the solutions $y(t)$ as $t \to \infty$.

2. Sketch the direction vector field for the equation $y' = y(2 - y)$. Use the direction field to describe the behavior of the solution as $t \to \infty$ for $y(0) = 1/2$ and $y(0) = 4$.

3. Solve the equation $y' = -1 - 2y$, $y(0) = 2$.

4. Solve the equation $y' = 3y + 1$, $y(0) = 1$.

5. A certain population satisfies the equation $\frac{dp}{dt} = 2p - 50$.
(a) Suppose $p(0) = 15$. When will the population become extinct.
(b) Suppose $p(0) = 30$. When will the population become extinct?

6. A spherical raindrop evaporates at a rate proportional to its surface area. Write down a differential equation for $V(t)$, the volume of the raindrop as a function of time t.

7. A falling object satisfies the equation

$$\frac{dv}{dt} = 9.8 - \frac{1}{5}v\,, \quad v(0) = 0\,.$$

(a) Find the time for the object to reach 98% of its limiting velocity.
(b) At the time found in (a) how far did the object fall?

8. A falling object (of mass m=10 kg) is submitted to negligible air friction. (Take $g = 9.8m/s^2$).
(a) Write down a differential equation for the velocity of the object and solve it.
(b) An object is dropped with initial velocity 0 from a height of 200 meter. Determine how long it take for the object to reach the ground.
(c) Determine the velocity at the time of impact.

9. A radioactive substance disintegrates at a rate proportional to the amount currently present. This means that if $Q(t)$ is the amount of substance at time t then $Q(t)$ satisifies the equation

$$\frac{dQ}{dt} = -rQ$$

where $r > 0$ is known as the *decay rate*.
(a) Solve the differential equation for arbitrary initial condition $Q(0)$.
(b) It is observed that after 1 day, the amount of a certain substance has changed from 10g to 6.2g. Determine the decay rate r.
(c) For the substance in (b), find the time until 0.5 g of the substance remains.
(d) The *half-life* τ of a radioactive substance is the time required for the an amount of a substance to decay to half of its original value. Find a relation between the half-life τ and the decay rate r.

10. A pound contains 10000 gal of water and an unknown of some chemical. Water containing $0.01g$ of this chemical per gallons flows into the pound at a rate of 300 gallons per hour. The mixture flows out at the same rate so the amount of water in the pound is constant. You should assume that water in the pound is well-mixed so that the concentration of chemical is uniform.
(a) Write down an differential equation for the amount of chemical, $S(t)$, in the pond at any given time. Take S in to be measures in grams and t in hours.
(b) After a very long time what is going to be the amount of chemical in the pound?
(c) Solve the differential equation found in (a).

Solutions:

1 (a) $y(t) \to 1/2$ as $t \to \infty$.

1 (b) $y(t)$ diverges as $t \to \infty$ except for the equilibrium solution $y(t) = -1/3$.

2 In both cases $y(t) \to 2$ as $t \to \infty$.

5 (a) $T = \frac{1}{2}\ln(2.5)$, (b) never.

6 $\dfrac{dV}{dt} = aV^{2/3}$.

8 (b) $t = \sqrt{200/4.9}$.

9 (b) r=0.478. (d) $r\tau = \ln(2)$

10 $\dfrac{dS}{dt} = 3 - \dfrac{3}{100}S$