Math 456: Homework 5

1. Find the stable matching produced by both the men proposing and women proposing algorithms for the following sets of preferences:

   (a) Three men called A, B, and C are to be matched with three women X, Y, and Z. Their preferences are

   $$\begin{array}{ll} A : X > Y > Z & X : C > B > A \\ B : Y > X > Z & Y : A > B > C \\ C : Y > X > Z & Z : C > A > B \end{array}$$

   (b) Four men called A, B, and C, and D are to be matched with four women W, X, Y, and Z.

   $$\begin{array}{ll} A : Y > X > Z > W & W : A > B > D > C \\ B : X > W > Y > Z & X : C > A > D > B \\ C : X > Z > W > Y & Y : C > B > D > A \\ D : Y > W > Z > X & Z : B > A > C > D \end{array}$$

2. Suppose that the gentlemen Albert, Bertram, Charles and David are matched with the ladies are Josephine, Katherine, Louise and Mary. Suppose their preferences are as follows.

   $$\begin{array}{ll} A : J > K > L > M & J : B > A > C > D \\ B : K > J > L > M & K : A > B > C > D \\ C : L > M > J > K & L : D > C > A > B \\ D : M > L > J > K & M : C > D > A > B \end{array}$$

   Show that there are at least 4 stable arrangements.

3. Consider the following two-player zero sum game. Both players simultaneously call out one of the numbers 2 or 3. Player R wins if the sum of the numbers called is odd and player C wins if their sum is even. The loser pays the winner the product of the two numbers called. Find the payoff matrix of the game, the value of the game and an optimal strategy for each player.

4. A zebra has four possible location to cross the Red River, call then a, b, c and d ordered from north to south. A crocodile can wait undetected at one of these locations. The crocodile will catch the zebra for sure if they choose the same spot (an earns a payoff of 1) and if zebra and crocodile choose adjacent location the crocodile will catch the zebra with probability 1/2 and so earns a payoff of 1/2.

(a) Write the 4×4 payoff matrix (for the crocodile as Row player) for the game.

(b) Use symmetries to reduce it to a 2×2 game.

(c) Find the value of the game and optimal strategies for the zebra and crocodile.

5. **(Non-transitive dice)** Consider the following set of three dice (each has six faces). The faces of the dice are marked with the following numbers

$$\begin{aligned} RED &: 3, 3, 3, 3, 3, 6 \\ BLUE &: 2, 2, 2, 5, 5, 5 \\ GREEN &; 1, 4, 4, 4, 4, 4 \end{aligned}$$

For each of the dice consider the random variable obtained by rolling the dice, e.g. $P(X_{RED} = 3) = 5/6$ and $P(X_{RED} = 6) = 1/6$ and so on....

(a) Show that

$$P(X_{RED} > X_{BLUE}) \quad P(X_{BLUE} > X_{GREEN}) \quad P(X_{GREEN} > X_{RED})$$

all greater than 1/2 hence the name non-transitive.

(b) Consider the following zero-sum game where each player selects one of the dice and then rolls it. The winner gets $1 from the loser. Write down the matrix of the (expected) gains.

(c) Find the optimal strategy for the game in (b). (Use the equality of payoffs theorem).

(d) Now change the game a little game by rolling *two* dice of the same color instead of one. What is the game matrix in this case? Are you surprised?

6. **(Non-transitive dice again)**

(a) Write down the game matrix for the following generalization of Rock-Paper-Scissor (of Big Bang Theory fame) known as Rock-Paper-Scissor-Spock-Lizard (see figure 1).

(b) The grime dice are the 5 dice of the form

$$\begin{aligned} Red &: 444449 \\ Yellow &: 333388 \\ Blue &: 222777 \\ Magenta &: 116666 \\ Olive &: 055555 \end{aligned}$$

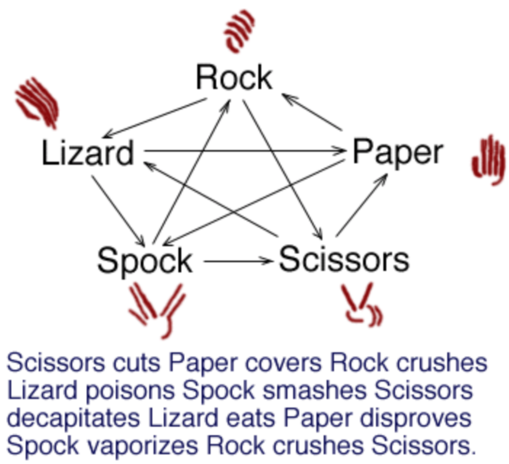
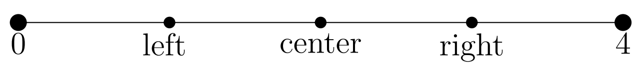


Figure 1: Rock-Papers-Scissors-Spock-Lizard

Consider the following zero-sum game where each player selects one of the dice and then rolls it. The winner gets \$1 from the loser. Write down the matrix of the (expected) gains. Can you guess an optimal strategy?

7. Consider the four mile stretch of road picture below



There are three locations along the road at which restaurants can be opened, $left$ at mile 1, *center* at mile 2, and *right* at mile 3. Company R wants to open 1 restaurant while company C wants to open 2 restaurants and several restaurants can occupy the same location. The customer are uniformly distributed along the road and one assume that they will go the nearest location with restaurant and choose the restaurant at random if there is more than one restaurant at that location.

We can take as payoff (gain for R) the probability that a customer chooses restaurant belonging to company R. For example if R open a restaurant a location $left$ and C opens restaurants at location $left$ and *center* the payoff is equal to $3/16$ since $3/8$ of the customers are closest to location $left$ and they will then choose the restaurant belonging to R with probability $1/2$.

Determine the value of the game and find some optimal (randomized) strategies for the companies.
Hint: Use domination to eliminate some strategies and write a 3 times 3 game matrix, you can also use symmetries.