Math 456: Homework 5

1. Recall that in the Powerball lottery you pick 5 out 69 numbers and 1 out 26 (the powerball). A ticket is worth \$ 2 and today on March 16 2019, the powerball jackpot is \$ 495 million with a cash value of \$ 300 million. If someone wins the jackpot, the next jackpot is reset to \$ 40 million (cash value around \$24 million).

   The prizes for the powerball are as follows

0+1	\$ 4
1+1	\$ 4
2+1	\$ 7
3+0	\$ 7
3+1	\$ 100
4+0	\$ 100
4+1	\$ 50,000
5+0	\$ 1,000,000
5+1	Jackpot

   (a) Compute the winning probabilities for all prizes.

   (b) Compute the expected value of a \$2 ticket without taking into account the jackpot.

   (c) What is the expected value (using jackpot cash values) of a \$2 ticket (a) today, (b) when the jackpot is reset, (c) when the jackpot hit the maximum of 1.5 billion (cash value around 930 million) in January 2016.

   (d) Use the Poisson approximation to compute how many tickets should be sold for the probability of having at least one winner to exceed (a) 1/2 and (b) .95.

2. Poker dice is a carnival game played with 5 dice with 9, 10, J, Q, K, A on the sides instead of the usual numbers. The bettor chooses two different faces (let us say he chooses Q and K) from the six choices and is paid at rate of 1:1 if both sides appear (that is they each appear at least once on the 5 dice). Compute your expected gain at this game. *Hint:* Compute the probability not to win and use the formula for $P(A \cup B)$.

3. Use the Poisson approximation to estimate that the probability that 3 people in a group of m people have the same birthday. How big should m be for this probability to exceed 1/2?

4. On August 18 1913, a roulette wheel in a casino in Monte Carlo, stopped 26 times in a row on black. What is the expected number of roulette runs needed until you see such an event.

5. You gather m people in a room and are interested in the event that all 365 birthdays are represented (ignoring leap years).

   (a) What is the expected number of people you should gather in a room so that all birthday are represented. *Hint:* Coupon collector problem.

   (b) What is the probability that a given birthday, say January 23, is not represented among the m people.

   (c) Use part (b) and a Poisson approximation to estimate the number of people needed to have a probability at least 1/2 that all birthday are represented in the room.

6. In both the Massachusetts Numbers Game and the New Hampshire Lottery, a four-digit number is drawn each evening from the sequence 0000, 0001, . . . , 9999. (This is one of the oldest form of lottery). On Tuesday September 9, 1981, the number 8902, was drawn in both lotteries. Lottery officials declared that the probability of both lotteries drawing the same number on that particular Tuesday evening was inconceivably small and was equal to one in one hundred million. Should you agree with this? Using the fact that the Massachusetts lottery was established in April 6 1972, and the New Hamphsire lottery in 1964 and a Poisson approximation estimate the the probability that both lotteries draw the same number at least once between April 6 1972 and Tuesday September 9, 1981?

7. Suppose you start with a fortune of 1 and throw a fair coin n times. A very rich and not very bright friend of yours pays you $\frac{11}{5}k$ for a bet of k if the throw is heads and nothing if the throw is tail.

   (a) Suppose you bet everything on each throw. What is your expected gain after 20 throws? What is the probability that you win nothing after 20 throws?

   (b) What does Kelly's formula suggest you should do in this situation? Determine the optimal proportion f and the optimal growth rate. What is your expected gain after 20 throws in that case?

8. Consider the proportional play strategy, but now every single bet of 1 unit leads to three possible outcomes

$$P(\text{Win } 4) = 1/2 \quad P(\text{Lose } 1) = 1/4 \quad P(\text{Lose } 4) = 1/4 \,.$$

   Is it a superfair bet? What is the optimal proportion f^* of your fortune you should invest and what is the optimal growth rate.

9. **(Kelly betting with interest)**.

(a) You are given the possibility to make a risky bet which returns with probability p, \$$\gamma$ for every dollar you bet. On the other hand if you do not bet you can invest your money in a bank and earn during the period of the bet an interest rate α. You decide to follow the proportional play strategy and invest a portion f of your money in the bet and to invest the remaining portion $(1-f)$ in the bank. Find the optimal f which will maximize the growth rate of your fortune.

(b) Suppose you are faced with a 100% safe investment returning 5% or a 90% safe investment returning 25%. Calculate how to invest your money using the Kelly strategy. Calculate the effective rate of return on your investment over the long term.