Math 456: Homework 4

1. Solve the following difference equations

   (a) $2x(n) - 5x(n-1) = 0$, $x(0) = 1$.

   (b) $x(n) - 5x(n-1) = 3$, $x(0) = 3$.

   (c) $2x(n) - x(n-1) = 2 + 3^n$, $x(0) = 0$.

   (d) $2x(n+1) - 7x(n) + 3x(n-1) = 0$, $x(0) = 1, x(1) = 2$.

   (e) $2x(n+1) - 7x(n) + 3x(n-1) = 2 + 2^n$, $x(0) = 3, x(1) = 0$.

2. Your mortgage is for 30 years with a fixed annual rate of $\alpha\%$ compounded monthly. Set-up and solve difference equations to solve the following problems

   (a) If you borrow \$150,000 today, what is the total amount of money will you pay back to the bank during the next 30 years if $\alpha = 2.75$ and $\alpha = 4.5$?

   (b) You can afford a down-payment of \$15,000 and a monthly payment of no more than \$1250. What is the value of the most expensive house can you afford if $\alpha = 2.75$ and $\alpha = 4.5$?

3. Your retirement account guarantees a fixed rate of 6% per year *paid yearly*. You start saving for retirement at age 25 with a target retirement age of 65 and \$0 in your saving account. Use first order difference equations to answer the following questions (all interests and payments are computed on a yearly basis).

   (a) You set aside \$500 every month (or \$6000 a year). How much money will you have for your retirement?

   (b) You want to retire with \$500,000. How much should you save every year?

   (c) You assume that your salary will increase 4% every year and so you decide your contribution should also increase by 4% every year. If your starting contribution is \$500 every month how much money will you have saved at retirement age?

   (d) Assuming again that your contribution is increasing by 4% every year, what should your starting contribution be if if you want to reach \$500,000 by retirement age?

4. As of today February 23 2019, the powerball jackpot is \$304 millions with a cash value of \$184.8 million. The cash value is the total amount of money you can get today if you win, while otherwise the jackpot of \$304 millions will be paid in 30 yearly installments of $\$304/30 = 10.13333...$ It turns out that virtually *every* winner decides to take the cash value of the jackpot. Are the winner making the right choice?

To decide what one should do assume that, instead of spending all your jackpot in a yacht and very expensive wines, you will invest your money and you can get guaranteed interest rate of $\%\alpha$ percent yearly (say compounded annually) and assume you invest and save all your money for 30 years.

(a) Write down two difference equations for each of the two options ($x(n)$ is the value of investment after n years in million dollars).

(b) For which interest rate α is the option of a lump sum better?

5. (a) At a certain casino card game which has paying odds of 1 to 1, your probability of winning each game is $p = 0.498$. You walk into the casino with \$25 dollars with the goal to get \$500. Compute the probability for you to succeed if you use the following strategies

   i. You make repeated \$5 bets until you either win \$500 or you are wiped out.

   ii. You make repeated \$25 bets until you either win \$500 or you are wiped out.

   iii. You play bold strategy.

   (b) By an extremely clever card counting trick you managed to change the odds of that game to $p = .502$. Compute again the probability that you succeed if you use the following strategies

   i. You make repeated \$5 bets until you either win \$500 or you are wiped out.

   ii. You make repeated \$25 bets until you either win \$500 or you are wiped out.

   iii. You play bold strategy.

6. Compute the bold strategy probabilities , $Q(1/7)$, $Q(2/7)$, $Q(3/7)$, $Q(4/7)$, $Q(5/7)$, and $Q(6/7)$.

7. (**Bolder is better**) At the game of roulette (Las Vega roulette with 38 numbers) you can bet on red/black with a payoff of 1 to 1 or you can bet on single number for a payoff of 35 to 1 (called a "straight up bet") or you can bet on a group of three numbers (called "street bet") with a payoff of 11 to 1.

   (a) Compute the expected gain (if you bet \$) for these three different bets.

   (b) After a long night, in circumstances best left untold, you "lost" a check of \$1,200 which was supposed to be used to pay your tuition bill. Walking in despair in front from a newly opened casino you find a crisp \$100 bill lying on the ground. You enter the casino and decide to bet until you have your \$1,200 back. Compute the probabilities to achieve this for

   i. Make repeated \$100 bets on red black.

ii. Use the bold strategy on red black.

iii. Use the bold strategy on a street bet.

iv. Divide your \$100 into three piles of \$33.333... each and make three straight up bets.

Discuss what is the best option? Why?

8. In this problem we are interested in how long a sequence of red/black bets will last until you go bust or win N. We denote

$y(j) =$ Expected number of bets until the game ends starting with a fortune of j

One can show (or guess!) that $y(j)$ satisfies the following second order equation

$$y(j) = 1 + py(j+1) + ry(j) + qy(j-1)$$

(a) What should be the "boundary conditions" $y(0)$ and $y(N)$?

(b) Find the general solution of the second order linear difference equation above. *Hint:* To find the particular solution try $y(j) = Aj + B$ for suitable constants A and B.

(c) Using (a) and (b) impute then expected number of bets until you go bust or reach a fortune of N starting with a fortune of j.

(d) Using (c) determine what the expected number of bets you can do starting with a fortune of j until you lose everything (you do no not stop if you fortune reach N but play for as long as you can).

(e) Compute the expected number of games played at the Las Vegas roulette and at the game of craps if your goals is to reach \$100 with bets of size 1 starting from (i) \$10 and (ii) \$50.