

Chapter 8: Stable matching and college admission

In this section we discuss the problem of matching a group (or groups) of people in pair, for example prospective students with schools, or men with women in such way that the matching are *stable*. The algorithm we describe is based on a famous paper by David Gale and Lloyd Shapley called "College admissions and the stability of marriage" [3]. The kind of algorithms described here is used, for example, to admissions to medical schools as well as in New York city [1] and Boston [2] public schools. This algorithm was discussed in the New York Times, see <https://nyti.ms/126O8RL>. The book by Alvin Roth [5] gives many application and examples of the stable matching problem and the book by Donald Knuth [4] analyzes the algorithm in depth and relates it to other well-known CS algorithms.

We will consider two problems

The kindergarten problem: Can we match $2n$ children in n stable pairs?

The marriage problem: Can we match n men and m women in stable pairs?

To make the problem we need to assume how preferences are made:

Assumptions on preferences: We assume that every participant has (strictly) ordered preferences for all possible matches and we also assume that one always preferred to be matched rather than not matched. If individual X decreasing order of preference is A, C, D, B we write

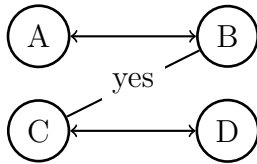
$$X : A > C > D > B$$

Definition of a stable matching: A matching M consists in forming pairs (of any two children or of one man with one woman). A matching M is called *unstable* if there exists two people who are not matched together but prefer each other to their partner in M . Otherwise we call the matching M *stable*.

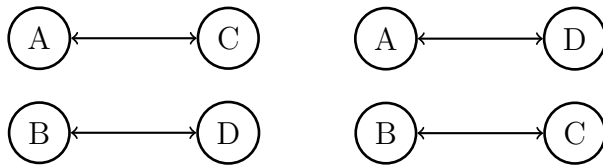
The kindergarten problem is not solvable: First we show that the kindergarten problem is in general not solvable (of course it depends on the choice of preferences) by providing a counterexample. Let us consider 4 children A, B, C, D with the following preferences

$$A : B > C > D \quad B : C > A > D \quad C : A > B > D \quad D : \text{any preference}$$

We claim that no matching is stable. Let us consider first the match



which is unstable since B prefers C to A and C prefers B to D . We leave it to the reader to verify that the matches



are also unstable. Therefore there are no stable matching for this particular choice of preferences.

Gale-Shapley Algorithm to find a stable matching for the marriage problem

Maybe surprisingly at first, the marriage problem is solvable and moreover there is an algorithm to find it! Its is called the *men-proposing algorithm*

- **Step 0** Initially all women are unmatched.
- **Step 1** Each man proposes to his most preferred woman who has not rejected him yet (or he gives up if he has been rejected by everyone).
- **Step 2** Each woman accepts tentatively a match with her most favorite proposer and rejects the rest.
- **Step 4** Repeat Step and Step 2 until there is no rejections.

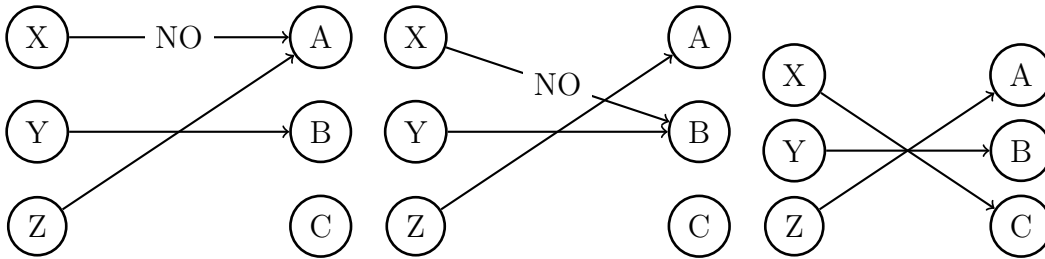
We could (probably should) of course have reversed roles and construct similarly a *women proposing algorithm*.

To see how the algorithm works let us consider an example of preference for three men X, Y, Z and three women A, B, C

$$\begin{array}{lll}
 X : A > B > C & Y : B > C > A & Z : A > C > B \\
 A : Y > Z > X & B : Y > Z > X & C : X > Y > Z
 \end{array}$$

The three rounds of proposal are depicted below. In the first round of proposals both X and Z proposes to A who rejects X because she prefers Z tot X . Then X moves to

his second choice B which rejects him because she prefers Y to X and finally X proposes his only remaining choice C .



The algorithm terminates here and one can check that this is stable: Y and B are each other first choices so no other matches can make any of them happier, X prefers A this his current match but A does not prefer X , and finally Z is matched to his first choice so he is happy.

This works for any choice of preferences.

Theorem: The men proposing algorithm leads to a stable matching.

Proof: The basic observation is that during the algorithm a man is tentatively matched to a sequence of women in decreasing order of preferences, starting with his most preferred match and then going down the list until the algorithm stops. A woman starts with her first proposal and then each round potentially is matched with a more preferred match if she can do some rejection.

First we show that the algorithm will stop after a finite number of steps. Indeed since in very round there is at least one rejection and by construction a woman will reject a man at most once. Therefore after at most $n \times m$ rounds the algorithm stops and ends up with a matching with $\min(n, m)$ pairs.

Next we show that when it terminates the matching obtained is stable. Let us assume that a man, BOB, and a woman, ALICE, are not matched to each other but BOB prefers Alice to his current match. Since BOB started with his most preferred match and went down the list. this means that ALICE rejected him at some point during the algorithm. But if ALICE rejected him this means she is currently matched to someone she prefers to BOB. Therefore BOB and ALICE do not create instability. Since the choice of BOB and ALICE were arbitrary this proves that the matching is stable. ■

Properties of the Gale-Shapley matching

We will show that among all possible stable matching the one given by the Gale-Shapley algorithm is actually *the best possible for men* and *the worst possible for women!* Think about this in the context of college/school admissions!

To state this properly we say that a *woman A is attainable for a man B* if there exists a stable matching M such that A is paired to B in the matching M

Theorem: In the Gale-Shapley matching M

1. Every man is matched to his most preferred attainable woman.
2. Every woman is matched to her least preferred attainable man.

Proof: For part 1. we argue by contradiction and assume that in the matching M some men are not matched to their most preferred attainable match. This means that at some point some of these men have been rejected by their most preferred attainable match. Let us look at the *first such rejection* where BOB was rejected by his most preferred attainable match ALICE who chose DAVID instead. Since this was the first such rejection it means that

DAVID likes ALICE at least as much than his most preferred attainable match

By definition there exists another stable matching \widehat{M} such that in \widehat{M} BOB and ALICE are matched and in \widehat{M} DAVID is matched to CLAIRE. But DAVID likes ALICE more than CLAIRE and ALICE prefers DAVID to BOB and therefore the matching \widehat{M} is unstable. This is a contradiction.

For part 2. we argue by contradiction again. Suppose that in the Gale-Shapley matching M ALICE is matched to BOB who is not her least preferred attainable match DAN. There is another match \widetilde{M} in which ALICE and DAN are matched and in this matching BOB is matched to, say, CAROL. By part 1. we know that in M BOB is matched to his most preferred attainable match and therefore BOB prefers ALICE to CAROL. But since ALICE prefers BOB to DAN then the matching \widetilde{M} is unstable. Contradiction! ■

If the number of men n is not equal to the number of women m then some people are left unmatched. We use the previous result to show that in any stable matching, the same people are matched

Corollary: The matched men and women are the same in all stable matching.

Proof: Assume that ALICE is unmatched in matching \widetilde{M} while she is matched to BOB in the Gale-Shapley matching. Since ALICE is BOB's most preferred attainable match, BOB prefers ALICE to whoever he is matched to in \widetilde{M} . Therefore \widetilde{M} is unstable. Contradiction!

In order to apply this algorithm to college/school admissions we note first that a college in generally will be matched to many students. Say if university X has 200 spots, you can think of of University X as 200 shadow universities X_1, \dots, X_{200} , each of which has the same ranking. Then we can assign preferences to the shadow universities to the students which reflects their true preferences.

References

- [1] Atila Abdulkadirođlu, Parag A. Pathak, and Alvin E. Roth. The new york city high school match. *American Economic Review*, 95(2):364–367, May 2005.
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- [4] Donald E. Knuth. *Stable marriage and its relation to other combinatorial problems*. American Mathematical Society, 1997.
- [5] Alvin E Roth. *Who Gets What and Why: The New Economics of Matchmaking and Market Design*. Eamon Dolan/Houghton Mifflin Harcourt, 2015.