Chapter 7: Proportional Play and the Kelly Betting System

Proportional Play and Kelly's criterion: Investing in the stock market is, in effect, making a series of bets. Contrary to bets in a casino though, one would generally believe that the stock market is on average *rising*, so we are making a series of *superfair bets*.

In order to make things concrete we imagine that we invest in the stock market by buying (idealized) *option contracts*. With such a contract you may pay a certain amount b to have the possibility to sell a certain stock at a certain price at a given date. If the market stock price is below your option price you will buy it on the market and sell it immediately at a profit. if the stock price is above your option price then you do nothing and you lose the amount you paid for your option. We will make it even simpler by assuming your option cost \$1 (this your bet), and you make a profit γ with probability p and lose your bet with probability $1-p$.

$$P(\text{ Win } \$\gamma) \;=\; p \quad P(\text{ Lose } \$1) \;=\; q = (1-p)$$

If the game is superfair then we assume

$$\text{Expected gain } E[W] \;=\; \gamma p - q > 0\,.$$

Informed with our experience with the gambler's ruin we recall that is subfair game the best strategy is *bold play*, that is invest everything at each step. But if the game is superfair then our probability to win a certain amount gets larger if we make small bets. But if we buy options that are sold on a fixed time window and making very small bets won't make you much money or only very slowly. To

Proportional Play = Invest a proportion $0 \le f \le 1$ of your fortune at each bet.

Let us know computed your fortune after n bets. If you start with a fortune X_0 then you bet fX_0. If you lose your fortune is $X_0(1-f)$ and if you win your fortune will be

$X_0(1-f)$ (what you did not bet) plus fX_0 (your original bet) plus $\gamma f X_0$ (your gain). So we have

$$X_1 = \begin{cases} X_0 + \gamma f X_0 & \text{if you win} \\ X_0 - f X_0 & \text{if you lose} \end{cases} = \begin{cases} X_0(1+\gamma f) & \text{with probability } p \\ X_0(1-f) & \text{with probability } q = 1-p \end{cases}$$

So at each step your fortune is *multiplied by a random factor* either $1+\gamma f$ or $1-f$, so we define the random variable

$$\boxed{Q = (1+\gamma f) \textbf{ with probability } p \textbf{ and } Q = (1-f) \textbf{ with probability } q\textbf{.}}$$

To find our fortune at time N we take

$$Q_1, Q_2, \cdots Q_N \text{ independent identically distributed random variables}$$

all with the same distribution as Q. Then we have

$$\boxed{\textbf{Fortune after } N \textbf{ bets is } X_N = Q_N Q_{N-1} \cdots Q_2 Q_1 X_0}$$

Since we make a series of super fair bets we expect that our fortune to fluctuate but to increase exponentially

$$X_N \sim X_0 \exp \alpha N \,.$$

and so α represents the rate at which our fortune increases. We can compute the long run value of α by using the law of large numbers. Indeed we have

$$\alpha \sim \frac{1}{N} \log(X_N/X_0) = \frac{1}{N} [\log(Q_1) + \log(Q_2) + \cdots \log(Q_N)]$$

and so the law of large numbers

$$\boxed{\frac{1}{N} [\log(Q_1) + \log(Q_2) + \cdots \log(Q_N)] \longrightarrow E[\log(Q)] \equiv \alpha}$$

To find the best asset allocation we seek to maximize $\alpha = E[\log(Q)]$.

$$\boxed{\textbf{Optimal proportional play} \iff \textbf{maximize } E[\log(Q)]}$$

We have

$$E[\log(Q)] \;=\; p\log(1+\gamma f) + q\log(1-f)$$

Differentiating with respect to f we find

$$\begin{aligned}
0 &= \frac{d}{df}E[\log(Q)] \\
&= \frac{d}{df}p\log(1+\gamma f) + q\log(1-f) \\
&= \frac{p\gamma}{1+\gamma f} - \frac{q}{1-f}
\end{aligned} \tag{1}$$

and so we have

$$p\gamma(1-f) = q(1+\gamma f)$$
$$p\gamma - q \;=\; f\gamma(p+q) \;=\; f\gamma$$

Finally we obtain

$$\boxed{\textbf{Optimal f}: \qquad \mathbf{f}^* = \frac{\mathbf{p}\gamma - \mathbf{q}}{\gamma} = \frac{\textbf{Expected gain}}{\textbf{gain}} \qquad \textbf{Kelly's formula}}$$

For example if your bet results in a payout of \$10 with probability 1/4 you should bet

$$f^* = \frac{10\frac{1}{4} - \frac{3}{4}}{10} = \frac{7}{40} = 0.175$$

of your fortune on each bet. Your fortune, on the long run, will grow at the rate of

$$\alpha = \frac{1}{4}\log(1+10\frac{7}{40}) + \frac{3}{4}\log(1-\frac{7}{40}) \;=\; 0.047$$

Kelly vs. Half-Kelly Let us consider a bet with $p = .0.4$ and $\gamma = 2$ so that we get the optimal

$$f^* = \frac{2\times 0.4 - 0.6}{2} = \frac{1}{10} \quad \text{and} \quad \alpha^* = 0.4\ln\left(1+2\frac{1}{10}\right) + 0.6\ln\left(1-\frac{1}{10}\right) = 0.0098$$

To get an idea why is going let us compute the rate of return for various value of α.

f	0.025	0.05	0.1	0.2	0.4
α	0.0043	0.0073	0.0098	0.0007	-0.07

What we see in this example is that vs respect to the Kelly rule ($f^* = 0.1$) using the so called Half-Kelly rule ($f = \frac{1}{2}f^* = 0.05$) reduces your rate return by a factor roughly equal to 3/4 why using using the double Kelly reduces your rate of return to essentially 0.

It is often said that playing the Kelly is too risky and that using half-Kelly is more safe. To understand this let us compute the variance of the rate of return

$$\sigma^2 = E[\ln(Q)^2] - E[\ln(Q)]^2 = 0.4\ln(1+2f)^2 + 0.6\ln(1-f)^2 - [0.4\ln(1+2f) + 0.6\ln(1-f)]^2$$

and we find that

$$f^* = 0.1 \sigma^2 = 0.0198$$
$$f^* = 0.05 \sigma^2 = 0.0051$$

From these number we learn that the variance is huge! If $f = 0.1$ then α is about 1 percent but the standard deviation of α is $\sqrt{0.00198}$ which is around 14 percent. This means that we should expect huge variation in our rate of return. We know that, eventually, we are going to be rich, but this is going to be a wild wild ride. Reducing f to the half-Kelly 0.05 reduce the standard deviation to $\sqrt{0.0051}$ which is about 7 percent. A bit less wild.

To get an idea on how wild it let us invest with the (modest) goal of doubling our money. We have

$$\ln\left(\frac{X_N}{X_0}\right) = \ln(Q_1) + \cdots + \ln(Q_N)\,.$$

and we are going to use the Central Limit Theorem to find N such that

$$P(X_N \geq 2X_0) \geq 0.95$$

We have

$$\begin{aligned}
P(X_N \geq 2X_0) &= P\left(\ln\frac{X_N}{X_0} \geq \ln(2)\right) \\
&= P\left(\ln(Q_1) + \cdots + \ln(Q_N) \geq \ln(2)\right) \\
&= P\left(\frac{\ln(Q_1) + \cdots + \ln(Q_N) - n\alpha}{\sigma\sqrt{n}} \geq \frac{\ln(2) - n\alpha}{\sigma\sqrt{n}}\right) \\
&\approx P\left(Z \geq \frac{\ln(2) - n\alpha}{\sigma\sqrt{n}}\right) \geq 0.95
\end{aligned}$$

where Z is the standard normal random variable. Since $P(Z \geq -1.65) =\approx 0.95$ we can find the corresponding n and find $n = 708$ if $f = 0.1$ but $n = 430$ of $f = 0.05$. With half-Kelly we have a lower rate of return but we need less time to double our money with some certainty.

Horse races: In the same spirit we can wonder what is the optimal strategy if we are betting on a horse race. We suppose there are m horses, and the betting odds are such that if we pay \$1 on on horse i then we earn γ_i of horse i wins. How should we bet to maximize our fortune in the long run? In order not to ever lose all our money it makes sense to split your money and bet on *every* horse. That is

Proportional Play = Bet a proportion b_i of your fortune on horse i.

In order to find the optimal proportion we assume that horse i has probability p_i to win. How do we find p_i is the interesting part of the problem. One possible way to estimate p_i is to trust the *wisdom of the crowd* and to use as p_i the proportion of players which bets on horse i. This is the idea behind *betting markets* where one assume that if many people bet real money on a certain outcome, then their collective knowledge will produce the "true" probability. This is a version of the *efficient market hypothesis* which asserts that stock always trade at their fair values. Or in the context of sport betting you may want to believe that a knowledgeable bettor will be able to find the right value of p_i by the virtue of his accumulated knowledge.

In any case, as in the Kelly betting system above we try to maximize the growth rate. If we bet b_i on horse i and wins we earn $\gamma_i b_i$ so if we start with a fortune X after betting our fortune will be $Q_b X$ where Q_b is a random variable with

$$P\{Q_b = b_i\gamma_i\} = p_i$$

and we need to find

$$\max_{b_i : \sum_{b_i}=1} E[\ln(Q_b)] \;=\; \max_{b_i : \sum_{b_i}=1} \sum_{i=1}^{m} p_i \ln(b_i\gamma_i)$$

This can be done using Lagrange multiplier. We set

$$F(b,\lambda) = \sum_{i=1}^{m} p_i \ln(b_i\gamma_i) + \lambda \sum_{i=1}^{m} b_i\,.$$

and differentiating with respect to b_i we find that the maximum should satisfy

$$p_i \frac{1}{b_i} + \lambda \;=\; 0 \qquad \text{or } b_i = -\frac{p_i}{\lambda}$$

Using that $\sum_i b_i = 1$ we find $\lambda = -1$ and thus

Optimal strategy is to bet a proportion $b_i = p_i$ of your fortune.

Asset allocation in the stock market: A similar situation occurs in the stock market. We suppose there are m stocks and the i^{th} stock rate of return is described by the random variable Z_i

$$Z_i = \text{ ratio of the value of stock } i \text{ tomorrow and the value of stock } i \text{ today}$$

that is $Z_i = .98$ it means the stock i lost two percent today.

To invest our money we decide to invest a proportion b_i of our fortune in stock i with $\sum_{i=1}^m b_i = 1$. With this allocation our fortune after N days will be

$$X_N = Q_N \cdots Q_1 X_0$$

where Q_i are IID copies of Q

$$Q = \sum_{i=1}^m b_i Z_i$$

and X_0 is the initial fortune.

In order to maximize the rate of growth of the fortune, we need to find

$$\max_{b_i \geq 0 \ \sum_{b_i} = 1} E[\log(\sum_{i=1}^m b_i Z_i)]$$