Chapter 10: General games

Part 1: Examples of games

We introduce here the basic objects involved in game theory. To specify a game ones gives

- The *players*.

- The set of all possible *strategies* for each player.

- The *payoffs*: if each player picks a certain strategy then each player receive a payoff represented by a number. The payoff for every player depends, in general, from the strategies of all other players. Payoffs have many different meanings, e.g., an amount of money, a number of years of happiness, the fitness in biology, etc... Our convention is that the highest payoff is deemed the most desirable one.

In the rest of this section we introduce some of the standard games in game theory, together with the "story behind them"

The prisoner's dilemma:. This is one of the most famous game in game theory. One possible story associated to it goes as follows. A prosecutor seeks two arrest two bank robbers. He lacks proofs of their involvement in the heist but has managed to have them arrested for a minor fraud charge. He offers to each them the following choices. If you confess but your accomplice does not then you will go free and your accomplice will be punished 8 years in jail. If you both confess then you will each get 6 years in jail. Finally, if none confess then they both will convicted for the minor fraud charge, say for 1 year in prison.

We will represent payoffs and strategies using a table. We will call the player Robert (R is for the "row" player) and Collin (C is for "column" player). For both Robert and Collin the strategies are confess or not confess.

		Collin	
		confess	not confess
Robert	confess	-6 -6	-8 0
	not confess	-6 -8	-1 -1

The rows represents the strategies of Robert, the columns the strategies of Collin. The numerical entries are understood as follows: in each box the lower left entry is the payoff for Robert while the the upper right entry is the payoff for Collin.

Solution of the prisoner's dilemma: To see what is the best option for the prisoner's dilemma one observes that if your accomplice confess then it is better for you to confess (8 years in jails) rather than not confessing (10 years in jail). On the other hand if your accomplice does not confess it is better for you to confess (0 years in jails) rather than not confessing (1 years in jail). Therefore the outcome of the game is that rational players will both confess and end in jail for 8 years. We will see later that the pairs of strategies "confess" and "confess" is an equilibrium for this game. The "dilemma" here is that if both do not confess they would get both 1 year in jail instead of 8 years, a much more preferable outcome for both! However without communication between the players, there is no mechanism to enforce this.

Duopoly: A strategic situation similar to the prisoner's dilemma occurs in many different contexts. Imagine for example two countries who produce oil and each can choose between producing 2 millions barrels/day or 4 millions barrels/day. The total output will be then 4, 6, or 8 millions barrels/day and the corresponding price will be $20, $12, $7 per barrel (a decreasing price reflects basic law of supply and demand). The payoff table is given by

		Country B	
		2 million	4 million
Country A	2 million	40 / 40	48 / 24
	4 million	24 / 48	28 / 28

Analyzing the game as in the prisoner's dilemma one finds that the best strategy is to produce $4 million per day because it leads to a higher revenue no matter what the other country does. This is of course in many ways a socially bad outcome since it leads to wasting a precious resource and lower revenue for the oil producing countries.

Battle of the sexes: Robert and Chelsea are planning an event of entertainment. Above all they value spending time together but Robert likes to go to the game while Chelsea prefers to go to the ballet. They both need to decide what to do tonight without communicating with each other. To produce a numerical outcome we assume that the value 1 is given to having your favorite entertainment while a value 2 is assigned to being together. This leads to

		Chelsea	
		game	ballet
Robert	game	2 3	1 1
	ballet	0 0	3 2

It is useful to relabel the strategies as "selfish" which means game for Robert and ballet for Chelsea and "altruistic" which means ballet for Robert and game for Chelsea. Then the game has a more symmetric structure and we have the payoffs s

		Chelsea	
		selfish	altruistic
Robert	selfish	1 1	2 3
	altruistic	3 2	0 0

Finding what to do is here more complicated. If both players are altruisitic, which is consistent with wanting to be together, then the outcome is the worst possible. So in this game being a bit selfish is good. However if both players are purely selfish the outcome is not that great. The best possible outcome occur if one is selfish and the other is altruistic. This outcome is better for both than the other options. We will think then of the combination of strategies "selfish" and "altruistic" as a solution for this game. However there are 2 such outcomes, and so how should you decide who is selfish?

Chicken game: This game was famous during the cold war since it captures the essence of an arm race. But you may find yourself many other applications of this game. Imagine two drivers racing toward each other at high speed on a very narrow road. Each driver has the option to swerve or to race on. If one swerves while the other races on he is ridiculed and called a chicken. If both swerves it is a tie and if none swerve it ends in mutual destruction

A payoff table consistent with this game is

		Collin	
		swerve	race on
Robert	swerve	1 1	2 0
	race on	0 2	-10 -10

We leave it to the reader to interpret these numbers and to analyze strategic situation the game.

Stag Hunt: Another classical game is played between two hunters. They can choose to go after either a stag or a hare. To hunt the stag (the bigger prize) both hunters need to go after him to catch him (i.e. they need to *cooperate* with each other). On the other hand they can decide to go alone and capture a hare which they can do without the help of the other hunters. We assume the prize for a stag is 8 (which they will share with the other hunter) while the prize for a hare is 3.

<div align="center">

Collin

		cooperate	alone
Robert	cooperate	4 / 4	3 / 0
	alone	0 / 3	3 / 3

</div>

Matching pennies: This children is game is played as follows. You have a penny that you can show either as head or tail. If both pennies coincide then Robert wins and takes Collin's penny while if they do not Collin wins and takes Robert's penny. The payoffs are given by

<div align="center">

Collin

		H	T
Robert	H	-1 / 1	1 / -1
	T	1 / -1	-1 / 1

</div>

Playing head or tail is not a winning strategy and it should be intuitively clear that the best option is is to play head or tail at random.

Rock-Paper-Scissor: This other well-known children game has three strategies and we have

Rock wins against scissor, scissor wins against paper, paper wins over rock

that is the strategies cyclically dominate each other. We shall encounter other situations where this structure occur but for now you may simply think of

Man eats chicken eats worm eats .....

A payoff matrix for the game is given by

		Collin		
		R	S	P
Robert	R	0 / 0	-1 / 1	1 / -1
	S	1 / -1	0 / 0	-1 / 1
	P	-1 / 1	1 / -1	0 / 0

and every child will tell you that the best way to play is to pick a random strategy.

Conclusion: After all these examples we want to develop a number of tools to analyze a game in a more systematic manner. This will center around the idea of an "equilibrium" which is a pair of outcomes where both players are as happy as possible, or rather as little unhappy as possible.... We shall also see many applications of these games in many situations, some maybe rather unexpected like in evolutionary biology where the players (animals) are not necessarily rational beings.

Part 2: Dominated strategies and their elimination

Let us consider a 2-player game, the players being named Robert (the Row player) and Collin (the Column player). Each of the players has a number of strategies at his disposal. In the examples in lecture 1 the number and the nature of strategies were the same for all players but this does not need to be so in general. We will use the notation s_R for a strategy for Robert and s_C for a strategy. The payoff structure is given in a table, for example,

		Collin		
		A	B	C
Robert	I	0 \ 5	4 \ 5	3 \ 0
	II	4 \ 0	3 \ 0	2 \ 5

where the lower left entry in each box is the payoff for Robert while the upper right entry is for Collin.

It will be useful to introduce two matrices, the matrix P_R which summarizes the payoff of Robert and the matrix P_C which summarizes the payoff of Collin. For the above example we have

$$P_R = \begin{array}{c} \\ I \\ II \end{array} \begin{array}{ccc} A & B & C \\ \left(\begin{array}{ccc} 5 & 5 & 0 \\ 0 & 0 & 5 \end{array}\right) \end{array} \qquad P_C = \begin{array}{c} \\ I \\ II \end{array} \begin{array}{ccc} A & B & C \\ \left(\begin{array}{ccc} 0 & 4 & 3 \\ 4 & 3 & 2 \end{array}\right) \end{array}$$

Definition 0.1. (Dominated strategy) *For a player a strategy s is dominated by strategy s′ if the payoff for playing strategy s′ is strictly greater than the payoff for playing s, no matter* what the strategies of the opponents are.

For the row player R the domination between strategies can be seen by comparing the rows of the matrices P_R. The strategy s dominate s' for R if $P_R(s, s_C) < P_R(s', s_C)$ for every s_C, that is every element in the row s of the matrix P_R is smaller than the corresponding entry in the row s':

$$P_R = \begin{array}{c} \\ s \\ \\ s' \\ \\ \end{array} \left(\begin{array}{cccc} \cdots & \cdots & & \\ a_1 & a_2 & \cdots & a_n \\ \cdots & \cdots & & \\ b_1 & b_2 & \cdots & b_n \\ \cdots & \cdots & & \end{array}\right) \qquad a_1 < b_1, a_2 < b_2, \cdots, a_n < b_n$$

For the column player C the domination between strategies can be seen by comparing the columns of of the matrices P_C. The strategy s dominate s' for C if $P_R(s_R, s) < P_R(s_R, s')$ for every s_R that is every element in the column s of the matrix P_C is smaller than the corresponding entry in the column s':

$$P_C = \begin{pmatrix} & s & & s' \\ & c_1 & & d_1 \\ \vdots & c_2 & \vdots & d_2 \\ \vdots & \vdots & \vdots & \vdots \\ & c_n & & d_n \end{pmatrix} \quad c_1 < d_1, c_2 < d_2, \cdots, c_n < d_n$$

Example: In our analysis of the prisonner's dilemma we have used the domination of strategy. The payoff matrix for Robert is

$$\begin{matrix} \text{confess} \\ \text{not confess} \end{matrix} \begin{pmatrix} -6 & 0 \\ -8 & -1 \end{pmatrix}$$

and clearly the strategy "confess" dominates the strategy "not confess". The situation is the same for both players and both should confess. ■

Principle of elimination of dominated strategies: If the players of the game are rational, then they should never use a dominated strategy since they can do better by picking another strategy, no matter what the other players are doing. In the same vein, if a player player thinks his opponents are rational, he will never assume that his opponent uses a dominated strategy against him. And a rational player knows that his opponent knows that he will never use a dominated strategy, and so on and on .....

The logical outcome is that a player will *never use dominated strategies*. So for practical purpose a dominated strategy never shows up in the game and we can safely discard it and play with a game with a smaller number of strategies. There is no reason why this process cannot sometimes be iterated leading to smaller and smaller games. This principle is called the *iterated elimination of dominated strategies*

Example: Solving a game by iterated elimination of dominated strategies Let us revisit the game given at the beginning of the lecture with payoff matrices

$$P_R = \begin{matrix} & a & b & c \\ I & \\ II & \end{matrix} \begin{pmatrix} 5 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad P_C = \begin{matrix} & a & b & c \\ I & \\ II & \end{matrix} \begin{pmatrix} 0 & 4 & 3 \\ 4 & 3 & 2 \end{pmatrix}$$

Inspection of the matrices shows that strategy b dominates strategy c for the column player and so we omit c and find the payoff matrices

$$P_R = \begin{matrix} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix} I \\ II \end{matrix} & \begin{pmatrix} 5 & 5 \\ 0 & 0 \end{pmatrix} \end{matrix} \qquad P_C = \begin{matrix} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix} I \\ II \end{matrix} & \begin{pmatrix} 0 & 4 \\ 4 & 3 \end{pmatrix} \end{matrix}$$

Now strategy I dominates II for the row player and so omitting II we have

$$P_R = \begin{matrix} & \begin{matrix} a & b \end{matrix} \\ I & \begin{pmatrix} 5 & 5 \end{pmatrix} \end{matrix} \qquad P_C = \begin{matrix} & \begin{matrix} a & b \end{matrix} \\ I & \begin{pmatrix} 0 & 4 \end{pmatrix} \end{matrix}$$

There is no more choice for the row player and we see that the best situation is for R to play I and for C to play b. ∎

Example: Consider the following game which has an arbitrary number of participants, say n of them. Each player announces a number between 1 and 100 and the winner is the one whose number is closest to half the average of these numbers, that is to the number

$$\frac{1}{2} \frac{k_1 + \cdots + k_n}{n}$$

For example for three players playing say, 1, 31, and 100, half the average is 22 and 30 will be the winner.

Before you read on you may want to make an experiment and play this game with your friends. What will they do? If you think about this game for a minute you may argue as follows: if everybody plays without thinking, the average will be around 50 and so I should play around 25. But if everybody makes the same argument the average will be 25 and so I should play around 12.5 and so on... This lead to argue that 1 is the dominant strategy.

This is correct but it is not true that 1 dominates every other strategy, in the previous example 1 is beaten by 31. But one can argue by eliminating iteratively dominated strategies. For example, irrespective of what the players do the average will be no more than 50 and so if you play say more than 50, say 99, then the distance between your number and the half the average is

$$99 - \frac{1}{2} \frac{99 + \cdots + k_n}{n}$$

Elementary algebra show then playing 99 is always better than playing hundred and so we can eliminate 100. You can repeat the argument and show that playing 1 is the rational strategy for this game.

You may want to try and play this game with your friends. Are they playing the dominating strategy? ∎

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Part 3: Nash equilibrium:

The mathematician John Nash introduced the concept of an *equilibrium* for a game, and equilibrium is often called a *Nash equilibrium*. They provide a way to identify reasonable outcomes when an easy argument based on domination (like in the prisoner's dilemma, see lecture 2) is not available.

We formulate the concept of an equilibrium for a two player game with respective payoff matrices P_R and P_C. We write $P_R(s, s')$ for the payoff for player R when R plays s and C plays s, this is simply the (s, s') entry the matrix P_R.

Definition of Nash equilibrium: A pair of strategies $(\hat{s}_R, \hat{s}_C)$ is a *Nash equilbrium* for a two player game if no player can improve his payoff by changing his strategy from his equilibrium strategy to another strategy provided his opponent keeps his equilibrium strategy.

In terms of the payoffs matrices this means that

$$P_R(s_R, \hat{s}_C) \leq P(\hat{s}_R, \hat{s}_C) \quad \text{for all } s_R\,,$$
$$P_C(\hat{s}_R, s_C) \leq P(\hat{s}_R, \hat{s}_C) \quad \text{for all } s_C\,.$$

The idea at work in the definition of Nash equilibrium deserves a name:

Definition of best response:

A strategy $\hat{s}_R$ is a *best-response* to a strategy s_c if

$$P_R(s_R, s_C) \leq P(\hat{s}_R, s_C) \quad \text{for all } s_R \quad \text{or} \quad \max_{s_R} P_R(s_R, s_C) = P(\hat{s}_R, s_C)$$

A strategy $\hat{s}_C$ is a *best-response* to a strategy s_R if

$$P_C(s_R, s_C) \leq P(s_R, \hat{s}_C) \quad \text{for all } s_C \quad \text{or} \quad \max_{s_C} P_C(s_R, s_C) = P(s_R, \hat{s}_C)$$

We can now reformulate the idea of a Nash equilibrium as

The pair $(\hat{s}_R, \hat{s}_C)$ is a *Nash equilibrium* if and only if $\hat{s}_R$ is a *best-response* to $\hat{s}_C$ and $\hat{s}_C$ is a *best-response* to $\hat{s}_R$.

Finding Nash equilibrium: A very simple procedure allows to identify the Nash equilibrium by inspecting the payoff matrices P_R and P_C. For $(\hat{s}_R, \hat{s}_C)$ we must have that $P_R(\hat{s}_R, \hat{s}_C)$ is a maximum of the entries on its column and $P_C(\hat{s}_R, \hat{s}_C)$ is a maximum of the entries on its row. This gives the easy algorithm

- Circle the maximum in each *column* of the matrix P_R.

- Circle the maximum in each *row* of the matrix P_C.

- If there is an entry (s_R, S_C) which is circled in *both* P_r and P_C then (s_R, S_C) is a Nash equilibrium.

Example: Consider the game with payoff matrices P_R and P_C given below. Circling the maxima on columns and rows we have

$$
P_R = \begin{array}{c} \\ I \\ II \end{array}
\begin{array}{ccc} A & B & C \\ \left(\begin{array}{ccc} \textcircled{5} & \textcircled{5} & 0 \\ 0 & 0 & \textcircled{5} \end{array}\right) \end{array}
\qquad
P_C = \begin{array}{c} \\ I \\ II \end{array}
\begin{array}{ccc} A & B & C \\ \left(\begin{array}{ccc} 0 & \textcircled{4} & 3 \\ \textcircled{4} & 3 & 2 \end{array}\right) \end{array}
$$

The entry corresponding to the pair of strategies (I, B) is circled in both matrices P_R and P_C and thus is a Nash equilibrium.

Example: Nash equilibrium for the prisoner's dilemma: We have

$$
P_R = \begin{array}{c} \\ C \\ N \end{array}
\begin{array}{cc} C & N \\ \left(\begin{array}{cc} \textcircled{-6} & \textcircled{0} \\ -8 & -1 \end{array}\right) \end{array}
\qquad
P_C = \begin{array}{c} \\ C \\ N \end{array}
\begin{array}{cc} C & N \\ \left(\begin{array}{cc} \textcircled{-6} & -8 \\ \textcircled{0} & -1 \end{array}\right) \end{array}
$$

and thus the Nash equilibrium is (C, C) as expected.

Example: Nash equilibrium in Battle of the sexes: We have

$$
P_R = \begin{array}{c} \\ S \\ A \end{array}
\begin{array}{cc} S & A \\ \left(\begin{array}{cc} 1 & \textcircled{3} \\ \textcircled{2} & 0 \end{array}\right) \end{array}
\qquad
P_C = \begin{array}{c} \\ S \\ A \end{array}
\begin{array}{cc} C & N \\ \left(\begin{array}{cc} 1 & \textcircled{2} \\ \textcircled{3} & 0 \end{array}\right) \end{array}
$$

and we have 2 Nash equilbria, namely (S, A) and (A, S).

Example: Nash equilibrium in the Stag Hunt: We have

$$
P_R = \begin{array}{c} \\ C \\ A \end{array}
\begin{array}{cc} C & A \\ \left(\begin{array}{cc} \fbox{4} & 0 \\ 3 & \fbox{3} \end{array}\right) \end{array}
\qquad
P_C = \begin{array}{c} \\ S \\ A \end{array}
\begin{array}{cc} C & N \\ \left(\begin{array}{cc} \fbox{4} & 3 \\ 0 & \fbox{3} \end{array}\right) \end{array}
$$

and we have 2 Nash equilbria, namely (C, C) and (A, A).

Example: Nash equilibrium in the Matching Pennies game:

$$
P_R = \begin{array}{c} \\ H \\ T \end{array}
\begin{array}{cc} H & T \\ \left(\begin{array}{cc} \fbox{1} & -1 \\ -1 & \fbox{1} \end{array}\right) \end{array}
\qquad
P_C = \begin{array}{c} \\ H \\ T \end{array}
\begin{array}{cc} H & T \\ \left(\begin{array}{cc} -1 & \fbox{1} \\ \fbox{1} & -1 \end{array}\right) \end{array}
$$

and we have no Nash equilbrium.

Pareto optimality. When applying game theory to social situation, think prisonner's dilemma or battle of the sexes, sometimes game theory yields to a outcome which seems not to be optimal from the point of view of social values. It sometimes seems like a "better" outcome which provides better payoffs to both players could occur. To quantify this we introduce the notion of *Pareto optimal* which is named after the economist Pareto.

Definition of Pareto optimal: A pair of strategies (s_R, s_C) in a two-player game, is *not Pareto optimal* is there exists another choice of strategies (s'_R, s'_C) such that both players are no worse off switching from (s_R, s_C) to (s'_R, s'_C) and at least one of the player is strictly better off (s_R, s_C) to (s'_R, s'_C). That is we have

$$
P_R(s'_R, s'_C) \geq P_R(s_R, s_C) \quad P_C(s'_R, s'_C) > P_R(s_R, s_C)\,,
$$

or

$$
P_R(s'_R, s'_C) > P_C(s_R, s_C) \quad P_R(s'_R, s'_C) \geq P_C(s_R, s_C)\,.
$$

For example the outcome predicted by game theory in the prisoner's dilemma is not Pareto optimal since by switching to not confess both players would be better off.

The fact that outcomes of a game are not always Pareto opimal should not be interpreted as a weakness of game theory. Sometimes being rational leads to socially destructive behavior and you can find plenty of such behavior in everyday life.

Symmetric game and social dilemma: Many of the examples in the previous lectures can be thought as describing social dilemma. These are games played by members of some group, who have the same incentives and interests although they will compete with each other.

We define a class of games which describe situation which are symmetric with respect to the players. In these games Robert and Collin are merely name (or label) assigned to players but the players are really interchangeable.

Definition 0.2. *A two player game is symmetric if*

- *The set of strategies is the same for the two players.*

- *The players are interchangeable, i.e. the payoff for R is R plays s and C plays s' is the same as the payoff for C if C plays s and R plays s'.*

In terms of the matrices P_R and P_C we have

> The game is **symmetric** if and only if P_R and P_C are transpose to each other ($P_R^T = P_C$)

Nash equilibria for two-player, two strategies, symmetric games: For a symmetric game with strategies 1 and 2 the general payoff matrices have the form

$$
P_R = \begin{array}{c} 1 \\ 2 \end{array}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad P_C = \begin{array}{c} 1 \\ 2 \end{array}\begin{pmatrix} a & c \\ b & d \end{pmatrix}
$$

Generically, if we exclude the cases where the some entries of the matrices coincide there are only three cases.

• **Case (1.1)** $\boxed{a > c \text{ and } b > d}$: There is one Nash equilibrium (1,1). The equilibrium is Pareto efficient if and only if $a > d$.

$$
P_R = \begin{array}{c} 1 \\ 2 \end{array}\begin{pmatrix} \textcircled{a} & \textcircled{b} \\ c & d \end{pmatrix} \qquad P_C = \begin{array}{c} 1 \\ 2 \end{array}\begin{pmatrix} \textcircled{a} & c \\ \textcircled{b} & d \end{pmatrix}
$$

Case (1.2) $\boxed{a < c \text{ and } b < d}$: There is one Nash equilibrium (2,2). The equilibrium is Pareto efficient if and only if $d > a$.

$$
P_R = \begin{array}{c} \\ 1 \\ 2 \end{array}
\begin{array}{cc} 1 & 2 \end{array}
\left(\begin{array}{cc} a & b \\ \textcircled{c} & \textcircled{d} \end{array}\right)
\qquad
P_C = \begin{array}{c} \\ 1 \\ 2 \end{array}
\begin{array}{cc} 1 & 2 \end{array}
\left(\begin{array}{cc} a & \textcircled{c} \\ b & \textcircled{d} \end{array}\right)
$$

The case (1.2) is really the same as (1.1) (just interchange the names of strategies..)

• **Case (2)** $\boxed{a > c \text{ and } b < d}$: There are 2 Nash equilibria $(1,1)$ and $(2,2)$ where the players pick the same strategy as their opponent. This type of game is called a *coordination game*.

$$
P_R = \begin{array}{c} \\ 1 \\ 2 \end{array}
\begin{array}{cc} 1 & 2 \end{array}
\left(\begin{array}{cc} \textcircled{a} & b \\ c & \textcircled{d} \end{array}\right)
\qquad
P_C = \begin{array}{c} \\ 1 \\ 2 \end{array}
\begin{array}{cc} 1 & 2 \end{array}
\left(\begin{array}{cc} \textcircled{a} & c \\ b & \textcircled{d} \end{array}\right)
$$

• **Case (3)** $\boxed{a < c \text{ and } b > d}$: There are 2 Nash equilbria $(1,2)$ and $(2,1)$ where the players pick the opposite strategy as their opponent.

$$
P_R = \begin{array}{c} \\ 1 \\ 2 \end{array}
\begin{array}{cc} 1 & 2 \end{array}
\left(\begin{array}{cc} a & \textcircled{b} \\ \textcircled{c} & d \end{array}\right)
\qquad
P_C = \begin{array}{c} \\ 1 \\ 2 \end{array}
\begin{array}{cc} 1 & 2 \end{array}
\left(\begin{array}{cc} a & \textcircled{c} \\ b & d \end{array}\right)
$$