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Nonlinearity from linearity: The Ermakov–Pinney equation revisited

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Abstract

In this short note, we revisit the so-called Ermakov–Pinney (EP) equation viewing its properties from a physically motivated perspective. We discuss its ties with the Schrödinger equation from such a perspective, demonstrating how the Ermakov–Pinney equation arises essentially due to the conservation of angular momentum. One of the main findings of the present work is how to use this conservation law to give a simple geometric proof of the nonlinear superposition principle applicable to the solutions of the EP equation. We also present ways in which the EP equation can be generalized and discuss their connections to earlier work. The other main novelty of this work consists of the generalization of the EP equation to higher dimensions.

Keywords: Angular momentum; Nonlinear superposition principle; EP equation

1. Introduction

The Ermakov–Pinney (EP) equation [5,14] is a quite famous example of a nonlinear ordinary differential equation (ODE). Such an equation (and generalizations thereof) have been shown to be relevant to a number of physical contexts including quantum cosmology [16], quantum field theory [6], nonlinear elasticity [19] and nonlinear optics [8], to name a few. A recent account of some of its properties along with applications in cosmological settings can be found in Ref. [9]. In particular, in the latter context the EP equation describes the temporal dynamics of an effective scale factor of the universe in simple cosmological models of the so-called Friedmann–Robertson–Walker type. The EP equation is also related to the so-called nonlinear Schrödinger equation describing, e.g., the wavefunction of Bose–Einstein condensates (BEC) at the mean-field level [13], or the envelope of the electric field in nonlinear optics [12]. In these contexts, the EP provides an effective description for the time-dependence of a moment of the relevant (spatially dependent) field, typically being associated with its width both in the BEC setting [7], as well as in the optical one [3]. A considerable amount of effort has also been placed in the mathematical analysis of the equation: see, e.g., [18] for a number of relevant generalizations or [17,11] for discretizations thereof that preserve some of its properties.

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Perhaps the most astonishing feature of this nonlinear equation is its direct connection with the linear Schrödinger (LS) equation. It is this connection that we would like to revisit in this short communication from a perspective motivated by physical arguments, and to show that this is an example where nonlinearity emerges (in an "integrable" way, as it should when this occurs) from linearity, due to the conservation law of angular momentum, or equivalently, as we discuss, of the Wronskian of the equation (in accordance with Abel's theorem). We should note that this picture was, in part, known already quite early [4]. However, here we complete this picture by demonstrating how the nonlinear superposition formula of the solutions to the EP equation geometrically stems from linearity and, in particular, from the (linear) superposition of the LS solutions. This is one of the main contributions of the present work. We also use these analogies to present a generalization of the EP equation to a setting respecting the conservation law, but where the nonlinearity is of arbitrary order (and also accompanied by nonlinear dispersion) as well as to a dissipative example, where the rotational symmetry is not respected. In both cases, the generalization of the superposition property is presented, ensuring (as it should) the integrability of these nonlinear (but stemming from linear) ODE's. Notice that these two generalizations have appeared in slightly different contexts elsewhere ([15] and [4], respectively); however, for reasons of completeness and of interpretation from a physical point of view, we believe that their discussion would be of interest to the reader. Finally, we also attempt two distinct generalizations of the equation in higher dimensions. One of them is particular to two-dimensions, while the other is relevant to a radially symmetric setting but independent of dimension. These connections/generalizations constitute the other main finding of the present study. Our physical perspective, together with the analogies and generalizations of the EP equation are presented in the next section while in the last section, we conclude and present some topics of potential interest for future work.

2. The EP equation and its generalizations

In order to obtain the nonlinear Ermakov–Pinney ordinary differential equation:

$$\rho_{xx} - \frac{L^2}{\rho^3} + a(x)\rho = 0, \tag{1}$$

(where the notation will be elucidated below), we start from the linear Schrödinger equation:

$$u_{xx} + a(x)u = 0. \tag{2}$$

In Eq. (2), u is the spatially (x) dependent field of interest and $a(x)$ is the external potential. x-Subscripts will be used throughout the manuscript to denote derivatives. Using a "polar-coordinate" representation of the field (in the plane) as $u = \rho\ \exp(\mathrm{i}\phi)$, Eq. (2) can be reshaped into the system:

$$\rho_{xx} - \rho\phi_x^2 + a(x)\rho = 0 \tag{3}$$

$$\frac{1}{\rho}(\rho^2\phi_x)_x = 0. \tag{4}$$

The second equation establishes the angular momentum (L) conservation law and can be rewritten as:

$$\phi_x = \frac{L}{\rho^2}. \tag{5}$$

Using, however, Eq. (5) in Eq. (3) produces a nonlinear ODE for the norm field

$$\rho_{xx} - \frac{L^2}{\rho^3} + a(x)\rho = 0, \tag{6}$$

with an "effective" (as it is usually called in classical mechanics) nonlinear potential $V(\rho) = L^2/(2\rho^2) + a(x)\rho^2/2$.

Hence, in this way, the *nonlinear* EP Eq. (6) has arisen from the *linear* LS Eq. (2). This is a prime example of linearity "effectively inducing" nonlinearity. It is therefore the conservation of angular momentum for planar, Hamiltonian dynamics which is effectively responsible for the essentially integrable behavior of the EP equation and its equivalence with the LS equation.

The connection between the two equations (thus the integrability of the EP equation) stems from the nonlinear transformation of u via its polar representation. The integrability of nonlinear equations, in particular PDEs, via a change of variables (of the dependent or independent variable) has been extensively discussed in Calogero's article in Ref. [2]. Such nonlinear equations are referred to as C-integrable equations. The analysis presented herein is similar in that a nonlinear transformation maps a linear ODE to a nonlinear ODE. It differs, however, from the work of Ref. [2] in the choice of the nonlinear transformation. The polar representation of the solution leads to a conservation law, thereby connecting the integrable behavior of the EP equation to the conservation law. Notice that the connections of the EP equation with the LS one through the angular momentum conservation law were, in part, discussed in Ref. [4]. However, we now extend these considerations to demonstrate how they yield the nonlinear superposition principle present in the EP setting.

Since the nonlinearity stems from linearity in this case (and can be transformed back to it), a complete "integrability" of the solution of the nonlinear problem should be expected; integrability here is meant in the sense that it should be possible to express the solutions of the nonlinear problem in closed form, given the solutions of the linear problem. This point is expressed in the following geometric picture/explanation which we believe elucidates the nonlinear superposition property of the EP equation. Suppose that two linearly independent solutions $u_1(x)$ and $u_2(x)$ are given. Then the *most general possible* solution of Eq. (2) can be written as $u = A_1u_1 + A_2u_2$. But then the *norm* of this solution in the plane (in accordance with the above transformation) will be a solution of the EP equation (see Fig. 1).

Then, if we represent $A_ju_j = \rho_j \exp(i\phi_j)$ $(j = 1, 2)$ and $\phi_{12} = \phi_1 - \phi_2$, the general solution ρ of the EP equation can be written as:

$$\rho = \sqrt{A_1^2u_1^2 + A_2^2u_2^2 + 2A_1A_2u_1u_2 \cos(\phi_{12})}, \tag{7}$$

which, in principle, has three free parameters. However, the conservation law of Eq. (5) can be used to rewrite this as the nonlinear superposition formula (with two free parameters) as follows. From figure

$$\phi = \arctan\left(\frac{A_2u_2 \sin(\phi_{12})}{A_1u_1 + A_2u_2 \cos(\phi_{12})}\right). \tag{8}$$

Using Eq. (8) in Eq. (5) and upon relevant simplifications we finally obtain:

$$A_1A_2W \sin(\phi_{12}) = L, \tag{9}$$

which is equivalent (upon substitution of ϕ_{12} from Eq. (9) into Eq. (7)) to the customary form of the nonlinear superposition formula (cf., e.g., with Eq. (13) of Ref. [9]). $W = u_1u_{2x} - u_{1x}u_2$ is the Wronskian of the LS equation.

The above polar representation lends itself to another interesting analogy. In particular, the Wronskian of the problem can be directly connected with the angular momentum [at least for linear (in u), Hamiltonian mechanics]. One way to see this analogy is by considering the linearly independent solutions $u_1 = \rho\cos(\phi)$ and $u_2 = \rho\sin(\phi)$ and form the Wronskian $W = \rho^2\phi_x \equiv L$. It should furthermore be highlighted that this analogy can be extended to parallelizing Abel's theorem for the conservation of the Wronskian in Eq. (2) with the conservation of angular momentum in this context. Eq. (9) constitutes the generalization of this connection (between W and L) for arbitrary solutions.

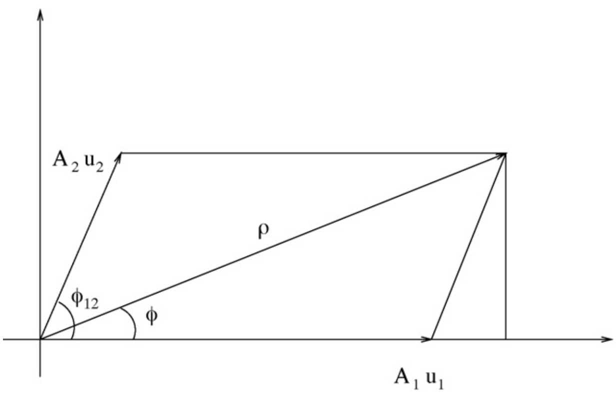

Fig. 1. Schematic illustration of the superposition principle in the plane: the norm of the general solution $u = A_1u_1 + A_2u_2$, emerging from the superposition of the solutions of the LS equation, will yield the general solution, ρ, of the EP equation.

Having presented the above physically motivated viewpoint on various aspects of the EP equation (such as the origin of its nonlinearity and of its superposition principle), let us now turn to two straightforward, one-dimensional but still interesting generalizations thereof.

A Hamiltonian, angular momentum conserving generalization can be obtained by using the alternative representation $u = \rho^k \exp(i\phi)$ instead of the standard polar form of $k = 1$. Using that and a procedure similar to the one employed previously (or substituting ρ by ρ^k in Eq. (6)), the following generalized equation is obtained:

$$\rho_{xx} + \mathcal{N}[\rho] + A(x)\rho = 0. \tag{10}$$

In the above equation $A(x) = a(x)/k$, while the nonlinear operator $\mathcal{N}$ acting on ρ is defined as:

$$\mathcal{N}[\rho] = \frac{k-1}{\rho}\rho_x^2 - \frac{\mathcal{L}^2}{\rho^{4k-1}}, \tag{11}$$

where the rescaled angular momentum satisfies $\mathcal{L}^2 = L^2/k$. Notice that Eq. (10) is the so-called Thomas equation [20], also examined in Refs. [15,10]. The nature of Eq. (11) (in conjunction with that of Eq. (10)) leads to some interesting conclusions. For the special case of $k = 1$, one immediately retrieves the standard EP equation. Since $4k - 1$ can take any value (some of the scaling relations such as the one for the angular momentum seem to imply that k should be positive but it is straightforward to observe that this restriction can be easily lifted), this transformation shows that the model can be made "integrable" for *any* nonlinear power. The price that one has to pay for that is that a *nonlinear dispersion* (see, e.g., the first term in the right hand side of Eq. (11)) has to be introduced. Still, this is a useful generalization of the example of "linearly induced nonlinearity"; it can, for instance, be used for a perturbative characterization of the nature of the solutions to the problem without the nonlinear dispersion starting from its "integrable", nonlinearly dispersive counterpart. Notice, by the way, that upon multiplication by $k\rho^{k-1}$, Eq. (10) can be brought in the immediately recognizable (as an EP generalization) form:

$$(\rho^k)_{xx} - \frac{L^2}{\rho^{3k}} + a(x)\rho^k = 0. \tag{12}$$

It should also be remarked that Eq. (10) bears an immediate generalization of the superposition principle through:

$$\rho = \left[A_1^2 u_1^2 + A_2^2 u_2^2 + 2A_1A_2u_1u_2 \cos(\phi_{12})\right]^{1/2k}, \tag{13}$$

We now turn to a different example of an EP equation, starting from a "dissipative" example of an ODE, namely

$$u_{xx} + b(x)u_x + a(x)u = 0, \tag{14}$$

where the first derivative plays the role of "dissipation" (of course, depending on its sign it can also play the role of a growth term). The polar representation goes through in this case also, but because of the dissipative nature of the model, the quantity $\rho^2\phi_x$ is no longer conserved. Instead substitution of the polar ansatz yields (upon solving a simple first order ODE)

$$\rho^2\phi_x = L = C \exp\left(-\int^x b(x')\,\mathrm{d}x'\right). \tag{15}$$

Eq. (15) lends further support to the analogy between the Wronskian and the (not conserved in this case) angular momentum. Using the generalized form of Abel's theorem [according to which $W_x = -b(x)W$], we immediately realize that the angular momentum and the Wronskian can be directly connected to (and in fact identified with) each other. We should note in passing that this "dissipative" version of the EP equation was mentioned in Ref. [4].

In this case, the resulting "dissipative EP" equation (written in a slightly more general form)

$$\rho_{xx} - \frac{f_x}{2f}\rho_x + a(x)\rho - \frac{f}{\rho^3} = 0, \tag{16}$$

where $f(x) = L^2(x)$, and L is given by Eq. (15). The regular EP equation falls under this setting for a constant f. In this case, the nonlinear superposition principle retains its form of Eq. (7); also, once again in Eq. (9) L/W will be a constant that determines the angle ϕ_{12} in Eq. (9). Eq. (16) may be rewritten in a form identifiable as an EP equation via rescaling the norm field

$$\psi = \rho \exp\left(\frac{1}{2}\int^{x} b(x')\,\mathrm{d}x'\right) \tag{17}$$

to obtain

$$\psi_{xx} - \frac{C^2}{\psi^3} + \left[a(x) - \frac{b_x}{2} - \frac{b^2}{4}\right]\psi = 0. \tag{18}$$

It is interesting to note that for $b(x) = 2/x$ (neglecting integration constants), Eq. (18) yields exactly, for ψ, the original form of the EP equation (cf. also with the special case $d = 3$ of Eq. (30) below).

As an example of a "dissipative EP" equation consider the motion of an inertial particle in a two-dimensional rotational shear. The carrier flow is, accordingly, $u = (-\alpha y, \alpha x)$ in a Cartesian (x, y) coordinate system with α being the shear rate; in polar (r, ϕ) coordinates, it becomes $u = (0, \alpha r)$. The dimensionless particle equations of motion, expressed in terms of the radial particle velocity ρ_t and the particle's angular momentum $L(t) = \rho^2\phi_t$, where the angular velocity is $\rho\phi_t$ and ϕ_t the angular frequency, are

$$\rho_{tt} - \frac{L^2(t)}{\rho^3} + \frac{1}{St}\rho_t = 0, \tag{19a}$$

$$L_t + \frac{1}{St}(L - \alpha\rho^2) = 0, \tag{19b}$$

The particle Stokes number, a dimensionless measure of particle inertia, has been denoted by St. These coupled, nonlinear equations are generalizations of Eqs. (15)–(16). The ODE associated with them is obtained by considering particle motion in the complex plane: if $u(t)$ is the particle position in the complex plane then

$$u_{tt} + \frac{u_t}{St} - \mathrm{i}\frac{\alpha}{St}u = 0, \tag{20}$$

an equation reminiscent of Eq. (14) except that the coefficient of $u(t)$ is purely imaginary. As described earlier, the linear derivative term in Eq. (20) may be eliminated by the rescaling $u(t) = f(t)\exp[-t/(2St)]$ to obtain

$$f_{tt} - \left(\frac{1}{4St^2} + \mathrm{i}\frac{\alpha}{St}\right) f = f_{tt} - \omega^2 f = 0. \tag{21}$$

As the coefficient of the u term in Eq. (20) is imaginary the coefficient of f in Eq. (21), the squared frequency ω^2, is complex. This has important consequences for the identification of the Wronskian with the angular momentum. Explicit differentiation of the Wronskian shows, as expected from Abel's theorem, that it satisfies a first order differential equation, $W_t = -W/St$. However, the particle angular momentum satisfies Eq. (19b), an inhomogeneous first order ODE. Comparison implies that the Wronskian and the angular momentum can not be directly connected when ω^2 in Eq. (21) is complex. The reason is that the two orthogonal and linearly independent solutions of Eq. (21) are no longer $\rho\cos(\phi)$ and $\rho\sin(\phi)$ as required and used earlier to find the relation between W and L, since ω^2 is complex.

The considerations of this exposition have been limited to one-dimensional cases up to this point. Furthermore, some of these examples were previously studied (e.g., in Refs. [4,15,20,10]), but we mention them here also to extend their physical interpretation and connect them with the EP equation proper. A first step towards expanding these considerations to higher dimensionalities highlights some further exciting analogies that we touch upon now.

Consider the simplest two-dimensional generalization of the linear Schrödinger equation, namely the one with a separable potential (which is effectively a one-dimensional problem):

$$\Delta u + V(x, y)u = 0, \tag{22}$$

where $V(x, y) = V_1(x) + V_2(y)$. If we now proceed to use the polar representation of the solution, the analog of Eq. (4) assumes the form:

$$(\rho^2\phi_x)_x + (\rho^2\phi_y)_y = 0. \tag{23}$$

Performing the customary separation of variables, using $\rho = f(x)g(y)$ and $\phi = \phi_1(x) + \phi_2(y)$ (notice that this is the appropriate choice to make the x- and y- dependent parts of Eq. (22) separable), we obtain the equations for f and g of the form:

$$f^2(\phi_1)_x - C\int^x f^2(x')\,\mathrm{d}x' = 0 \tag{24}$$

$$g^2(\phi_2)_y + C\int^y g^2(y')\,\mathrm{d}y' = 0 \tag{25}$$

$$f_{xx} - \frac{[C\int^x f^2(x')\,\mathrm{d}x']^2}{f^3} + [V_1(x) - K]f = 0 \tag{26}$$

$$g_{yy} - \frac{[C\int^y g^2(y')\,\mathrm{d}y']^2}{g^3} + [V_2(y) + K]g = 0, \tag{27}$$

where C, K are appropriate integration constants. Notice the integro-differential nature of the resulting equations, a feature absent in the one-dimensional problem. However, as in the one-dimensional case, here as well the solutions of Eqs. (24)–(27) can be directly related to the solutions of the linear Eq. (22).

As another example of a higher dimensional generalization of the EP equation, we consider the linear Schrödinger equation with a spherically symmetric d-dimensional potential $V(r)$. The decomposition of the solution into a radial factor $u(r)$ times a d-dimensional spherical harmonic leads to

$$u_{rr} + \frac{d-1}{r}u_r - \frac{l(l+d-2)}{r^2}u + V(r)u = 0, \tag{28}$$

where $l(l+d-2)/r^2$ is the d-dimensional rotational "energy" of a state with angular momentum l. Thus, the radial equation becomes a realization of the "dissipative" example Eq. (14) with $b(x) = (d-1)/x$. Following the steps that lead to Eqs. (16) and (18), we obtain the analog of Eq. (6)

$$\chi_{rr} - \frac{C^2}{\chi^3} + V_{\mathrm{eff}}(r)\chi = 0, \tag{29}$$

where

$$V_{\mathrm{eff}}(r) = V(r) - \frac{l(l+d-2)}{r^2} - \frac{(d-1)(d-3)}{4r^2}, \tag{30}$$

for the rescaled norm field [cf. Eq. (17)]

$$\chi = \rho\, r^{1/2(d-1)}. \tag{31}$$

The quantity C is given by Eq. (15) for the specific form of $b(x)$, namely $\rho^2\phi_r = C/r^{d-1}$.

3. Conclusions

In this short communication, we presented the Ermakov–Pinney equation from a physically oriented viewpoint, tracing the origin of its nonlinearity back to the conservation of angular momentum. The polar representation was useful not only in establishing the well-known connection of the Ermakov–Pinney equation to the linear Schrödinger equation, but also in reexamining from a geometric standpoint its nonlinear superposition principle. In the light of this perspective, the connections between the angular momentum and the Wronskian, and the conservation of the former with the Abel theorem concerning the latter were also mentioned. These considerations also led us to discuss possible generalizations of this equation and to continue along the vein of linearly induced nonlinearity, for general powers of the nonlinearity. We saw that the price that one has to pay for this generalization is the introduction of nonlinearly dispersive terms. We also revisited a "dissipative" generalization in which the angular momentum is no longer conserved, and generalized the relevant connections in that case. Finally, we examined the variation in the form of the equation for simple higher-dimensional setups (a separable two-dimensional case, as well as a spherically symmetric d-dimensional case).

It is interesting to note that this program of nonlinearity originating from linearity would not be realizable if the setting of Newton's (or, alternatively, Schrödinger's) equation involved third (or higher) derivatives. It would be interesting to examine whether this program can systematically be carried through in different settings (for instance, ones of higher dimensions) and what its outcome would be. For most of the above exposition, one-dimensionality of the underlying system was a rather crucial constraint. It would be very desirable to extend such considerations systematically to higher dimensions.

It is also worth mentioning that the origin of the nonlinearity, arising from a linear system, can be traced in the nonlinear transformation used in the solution ansatz (i.e., in the polar representation). In that sense, parallels can be drawn between this example and the well known case of the Burgers equation, which is also linearizable through a nonlinear transformation (the so-called Cole-Hopf transformation; see, e.g., [1]). Similarly, the generation of nonlinear equations from linear equations via a change-of-variables transformation (C-integrable equations) has been discussed in Ref. [2].

It is, thus, worth examining whether other, appropriately crafted, nonlinear transformations of linear problems can give further insight into (physically relevant) nonlinear equations. Such studies will be left for future investigations.

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