Homework Problems

1) Consider the model

$$u_{tt} = u_{xx} - \sin(u) + \epsilon\delta(x)\sin(u)$$

Often to understand the dynamics of such perturbed models for wave solutions, we use the variational or collective coordinate method. That is we use the waves of the unperturbed equation in the Lagrangian (or Hamiltonian) of the perturbed ones to determine how parameters of the solution (such as its center) are affected by the perturbed dynamics. More specifically:

1. Write the Lagrangian of the perturbed model.
2. To consider kink dynamics, use $u = 4\arctan(\exp(x - X(t)))$ in the Lagrangian and integrate over x to obtain a 1-degree-of-freedom *effective* Lagrangian which depends on $X(t)$ and $\dot{X}(t)$.
3. Set up the Euler-Lagrange equation for $X(t)$ and try to integrate the quadrature, assuming that $\dot{X} = v$, for $X \to -\infty$.

2) Consider the model

$$i\dot{u}_{n,m} = -C(u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1} - 4u_{n,m}) - |u_{n,m}|^2 u_{n,m}$$

1. Write down the energy and the (l^2) norm that should be conserved quantities for the dynamics and illustrate that the l^2 norm is conserved (similarly but with more tedious calculations the energy can be shown to be conserved).
2. Upon using $u_{n,m} = e^{it}v_{n,m}$, build a Newton code, say on a 21×21 grid, to find the following solutions:
   - a quadrupole with $v_{-1,-1} = v_{1,1} = 1$ and $v_{1,-1} = v_{-1,1} = -1$ and
   - a vortex with $v_{0,0} = 1$, $v_{1,0} = i$, $v_{1,1} = -1$ and $v_{0,1} = -i$.

   Perform a continuation of these solutions from $C = 0$ to $C = 0.1$. Together with the continuation, perform a linearization around the relevant solutions and solve the ensuing eigenvalue problem. In particular, show the dependence on C of the eigenvalue pairs that are 0 at $C = 0$. Append also to your homework your Newton and linear stability codes.