

## Homework Problems

1) Consider the model

$$u_{tt} = u_{xx} + 2(u - u^3)$$

1. For traveling wave solutions  $u(x, t) = f(x - ct)$  with  $c^2 < 1$ , set up the relevant ODE and find the corresponding potential, and phase plane representation of the solutions.
2. For different “energies”, construct the corresponding phase plane curves, *as well as* the real space profiles of the solutions.
3. Identify the energy for which there is a heteroclinic orbit and integrate the quadrature problem to obtain it in an explicit analytical form.
4. Finally, obtain the explicit form of *bounded* solutions with “energy” less than that of the heteroclinic orbit [show also that the heteroclinic orbit naturally arises as a special case limit of these solutions].

2) Consider the model

$$u_{tt} = u_{xx} - \sin(u)$$

1. For traveling wave solutions  $u(x, t) = f(x - ct)$  with  $c^2 < 1$ , set up the relevant ODE and find the corresponding potential, and phase plane representation of the solutions.
2. For different “energies”, construct the corresponding phase plane curves, *as well as* the real space profiles of the solutions.
3. Identify the energy for which there is a heteroclinic orbit and integrate the quadrature problem to obtain it in an explicit analytical form.

3) More generally for models of the form:

$$u_{tt} = u_{xx} - V'(u)$$

where  $V(u)$  is a general potential, show that the PDE is invariant under a Lorentz transformation of the form:  $\tilde{x} = \gamma(x - vt)$  and  $\tilde{t} = \gamma(t - (v/c^2)x)$ ,

provided that  $\gamma$  has an appropriate form (which you should find). [Notice that this implies that if we find stationary solutions in these models we can boost them to arbitrary speeds  $v$  such that  $v^2 \leq c^2$ ].

4) For the diffusion equation, identify the exponents  $m$  and  $n$ , under which self-similar solutions of the form  $u(x, t) = t^m f(xt^n)$  can be obtained. In the case of  $m = 0$ , find these solutions in an explicit analytical form.

### Practice Problems

1) For the viscous Burgers equation

$$u_t + uu_x = \nu u_{xx}$$

identify the exponents  $m$  and  $n$ , under which self-similar solutions of the form  $u(x, t) = t^m f(xt^n)$  can be obtained. Write down the resulting ODE for  $f = f(\eta) = f(xt^n)$ .

2) Find the linearization PDE around a solution of the equation

$$u_{tt} = u_{xx} - \sin(u)$$

Identify the eigenvalues of the linearization around the steady states  $u_0 = 0$  and  $u_0 = \pi$ .