Homework Problems

1) Consider the nonlinear dynamical lattice equation

$$i\dot{u}_n = -C(u_{n+1} + u_{n-1} - 2u_n) - |u_n|^2 u_n$$

i.e., the discrete NLS model.

1. Solve analytically the limit $C = 0$ (the anti-continuum limit). *Hint*: do notice the conservation law that simplifies the relevant integration.

2. Motivated by the above limit, use $u_n(t) = e^{i\mu t} v_n$ and write down the equation for the v_n's.

3. Now set $\mu = 1$, and assume that the v_n's are real, for simplicity. At $C = 0$, form a configuration where all sites are 0 except for 1 nonzero site, form another where all sites are zero except for 2 sites (say, both of amplitude 1) and similarly one with 3 excited sites (again all of amplitude 1). Perform the continuation of all 3 of these configurations via Newton up from $C = 0$ up to $C = 0.1$. Also perform the linearization around the solutions and include it in the code that does the Newton (after every Newton step). Attach one example of such a continuation code.

4. You should convince yourselves that the 1-site mode is always stable (in fact for *any* C). On the other hand, you should see that the 2-site mode has one unstable eigenvalue and the 3-site mode has two unstable eigenvalues. Plot these eigenvalues as a function of C and compare them with the prediction of $2\sqrt{\epsilon}$ (for the 2-site) and $\sqrt{2\epsilon}$ and $\sqrt{6\epsilon}$, given theoretically in section 4 of Physica D **212**, 1-19 (2005).

5. Plot the two smallest eigenvalues of the operators L_+ and L_- for the 2-site case, and also the three smallest eigenvalues for the 3-site case. Can you theoretically justify that L_- always seems to have a 0 eigenvalue ?

6. Find the profile of the unstable 2-site mode at $C = 0.1$, and use it as initial condition for a 4th-order Runge-Kutta integrator that will evolve this solution as a function of time, up to, say, $t = 300$. Using the command `imagesc`, plot a space-time graph of the evolution of $|u_n(t)|^2$. Can you find a way to plot the evolution of the center of mass of the solution vs. time ? Notice that you may have to add a small perturbation to the solution to initiate the instability.

2) Similarly analyze the stability of the 2-site solution of the saturable DNLS model

$$i\dot{u}_n = -C(u_{n+1} + u_{n-1} - 2u_n) + u_n/(1 + |u_n|^2).$$

- I.e., Solve it in the AC limit, perform a continuation of the 2-site steady state (say, for $\mu = -0.5$) and find its real eigenvalue as a function of the coupling C, for small values of C (say, up to $C = 0.01$). Can you predict a theoretical relation that approximates the dependence of the relevant eigenvalue on C ?

- Now continue this solution further – what happens qualitatively (regarding the stability of this solution) at larger values of C ? What happens correspondingly to 1-site solutions ? *Hint*: Check what happens e.g. close to $C = 0.25$ to each of these solutions in terms of stability.

Practice Problems

1) Find the choice of coefficients of the 2nd order Runge-Kutta method that eliminates as many as possible of the error terms of $O(h^2)$.

2) Find the linear spectrum around the uniform steady states of the discrete ϕ^4 model:

$$\ddot{u}_n = C\Delta_2 u_n + 2(u_n - u_n^3)$$

3) Show that the operator L_- in the case of NLS models has a zero eigenvalue by identifying a relevant eigenvector with such an eigenvalue.

4) Show that the Hamiltonian $\sum_n |u_{n+1} - u_n|^2 - |u_n|^4$ is conserved in the discrete NLS model.