Homework Problems

1) For the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q},$$

show the Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0.$$

2) For the ϕ^4 model

$$u_{tt} = u_{xx} + 2(u - u^3)$$

identify the Lagrangian (showing that the Euler-Lagrange equations yield the above PDE), perform the Legendre transform to obtain the relevant Hamiltonian. Identify (through Noether's theorem) the conservation laws that correspond to the invariance of the Lagrangian with respect to space and time translations and prove that the corresponding conserved quantities are indeed conserved !

3) Show that the first 3 integrals of the KdV, namely the mass, momentum and the energy are mutually in involution, based on the Poisson bracket structure that we defined for the KdV. [This is a prerequisite for integrable systems, namely that the integrals of the motion are in involution]. Also show that the KdV is a bi-Hamiltonian model, namely that it has a second Hamiltonian $H_2 = (1/2) \int u^2 dx$, but with the Poisson operator

$$B = \partial_{xxx} + (1/3)(u\partial_x + \partial_x u)$$

[The bi-Hamiltonian nature is a property that implies the existence of infinite conservation laws which are in involution under both types of Poisson brackets].

4) Starting from the KdV $u_t + uu_x + u_{xxx} = 0$, show that:

- if v defined by $u = -6(v^2 + v_x)$ satisfies the MKdV

$$v_t - 6v^2 v_x + v_{xxx} = 0,$$

  then u indeed satisfies the KdV.

- Show that the KdV is invariant under the transformation $x' = x + t/\epsilon^2$, $t' = t$, $u = u' - 1/\epsilon^2$, i.e., that u' still satisfies the KdV in the new variables.

- Show that if $v = -\epsilon w(x', t') + \frac{\beta}{\epsilon}$, where $\beta = 1/\sqrt{6}$ and x', t' are defined as above, then

$$w_{t'} + \partial_{x'}(w_{x'x'} + \sqrt{6}w^2 - 2\epsilon^2 w^3) = 0$$

  Notice that this directly implies that $(d/dt') \int w dx' = 0$.

- Now use the two transformations above together to find an expression for u' as a function of w.

- Finally, expand $w = \sum_{n=0}^{\infty} \epsilon^n w_n$ and use it in the above expression for u' to obtain the connection between the various orders w_n and u.

- Compute the first few w_n's and, if possible, a recursion relation that yields w_n. What does the conservation of $\int w dx'$ imply, order by order ?

Practice Problems

1) A head start into the last problem: Can you prove that $u = -6(v^2 + v_x)$ (so-called Miura transform) satisfies the MKdV

$$v_t - 6v^2 v_x + v_{xxx} = 0,$$

then u satisfies the KdV ?

2) Can you use quadratures to compute the analytical expression for the soliton of the MKdV ?