Homework Problems

1) Consider the nonlinear dynamical lattice equation

$$\ddot{u}_n = K(u_{n+1} + u_{n-1} - 2u_n) - \omega_0^2 u_n + \beta u_n^3$$

1. Assuming a carrier wave of frequency ω_0, look for a new dynamical lattice equation describing a slow envelope around it, in the form

   $$u_n = \phi_n e^{-i\omega_0 t} + \phi_n^\star e^{i\omega_0 t}$$

   Assume that: $\dot{\phi}_n \ll \omega_0 \phi_n$ (i.e. the envelope varies slowly with respect to the oscillation with frequency ω_0) and also that higher harmonics (such as $e^{3i\omega_0 t}$) can be neglected (this is the so-called "rotating wave approximation).

2. Using the new dynamical lattice equation obtained, consider solutions $\phi_n = \phi_0 e^{i(qn - wt)}$ and find under what conditions they satisfy the equation.

3. Finally, perform a modulational stability analysis of these solutions (by perturbing both their amplitude and phase), as we did in the continuum case. Do you observe any fundamental differences in the resulting stability equation with respect to its continuum analog obtained in class [hint: consider the defocusing case of $\beta < 0$] ?

2) Consider the NLS equation with saturation ($S > 0$):

$$iu_t + \frac{1}{2}u_{xx} + \frac{u|u|^2}{1 + S|u|^2} = 0$$

1. Seek solutions $u(x,t) = f(x)e^{i\omega t}$, with $\omega > 0$ and write down the ODE for $f(x)$.

2. Write down the solution of the ODE by quadrature (do not attempt to evaluate the integral).

3. Considering the relevant potential, identify a condition so that it can support homoclinic orbits. Under that condition, sketch various orbits of the system and their corresponding phase space representation.

3) Consider the generalization of the NLS

$$iu_t + \frac{1}{2}u_{xx} + su\int_{-\infty}^{\infty} R(x-x')|u(x',t)|^2 dx' = 0$$

where the kernel R is normalized as $\int_{-\infty}^{\infty} R(x) = 1$.

1. Find the dispersion relation $\omega_0 = \omega_0(k_0)$ for the plane waves $u(x,t) = \sqrt{\rho_0}e^{i(k_0x-\omega_0 t)}$.
2. Perform the linear stability analysis around the plane wave. For simplicity allow only for perturbations of the amplitude i.e.,

   $$u(x,t) = \left[\sqrt{\rho_0} + a(x,t)\right] e^{i(k_0x-\omega_0 t)}$$

   For simplicity, once you write the equation for $a(x,t)$ make the change of variables $\tau = t$ and $\xi = x - k_0t$ and write the linear PDE for $a(\xi,\tau)$.
3. Solve the ensuing linear equations [Hint: use Fourier transforms] and find the condition for linear stability of the plane wave.
4. Retrieve the local limit (that we did in class), by an appropriate selection of the kernel.
5. Consider the kernel $R(x) = 1/(2\sigma)$ for $|x| \le \sigma$ and $R(x) = 0$ for $|x| > \sigma$. Is it possible to obtain modulational instability even in the defocusing case of $s = -1$?

Practice Problems

1) For the NLS equation $iu_t = -(1/2)u_{xx} + g|u|^2u$, show that the energy $H = \int (1/2)[|u_x|^2 + g|u|^4]dx$ is a conserved quantity.

2) For the NLS $iu_t = -u_{xx} + (1+\cos(\omega t))|u|^2u$, find the spatially homogeneous solution $u_{hom}(t)$. Then use $u(x,t) = u_{hom}(t)[1 + \epsilon w(t)\cos(kx)]$ to find (to O(ϵ)) an ODE describing the time evolution of the perturbations $w(t)$.