Homework Problems

1) Consider the PDE

$$xu_t + uu_x = 0.$$

Upon a simple transformation that makes it look like an equation we have studied, solve it with initial conditions $u(x,t) = x$.

2) Consider the KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0.$$

- Let us look for traveling wave solutions $u(x,t) = f(x - ct)$. Write the resulting ODE for $f = f(\xi)$ where $\xi = x - ct$.
- Integrate this equation once, assuming that both f and its derivatives vanish at $x \to \pm\infty$.
- Write the resulting ODE as $f'' = \dots$ and solve it via quadrature, to find the prototypical KdV solution that decays (and has decaying derivatives) at $\pm\infty$.

3) Still for the KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0.$$

By passing the derivative inside the integral as a partial derivative, and assuming that u and its derivatives vanish sufficiently fast at $\pm\infty$, show that the quantities

- $M = \int_{-\infty}^{\infty} u dx$,
- $P = \int_{-\infty}^{\infty} u^2 dx$
- $E = \int_{-\infty}^{\infty} u^3 - \frac{(u_x)^2}{2} dx$,

are conserved quantities, i.e., that their time derivatives vanish. Interestingly, as we'll discuss further below, the KdV has infinitely many such conserved quantities, of these are only the first (and more physical) few. By the way, notice that it sounds plausible that the mass is defined as above (area under the field profile). However, illustrate that it also makes sense to define the momentum as above, by showing that if we define the center of mass of the field distribution as

$$x_c = \int_{-\infty}^{\infty} xudx/M,$$

then

$$\frac{d}{dt}(Mx_c) = 3P$$

which resembles the standard definition of the momentum (up to a constant prefactor).

4) Another variant of the PDE $u_t + uu_x = 0$ is its viscous version (so-called viscous Burgers' equation), namely:

$$u_t + uu_x = \nu u_{xx}.$$

- For this equation, show that if we start from a solution of the heat equation $s_t = \nu s_{xx}$ and make the transformation $u(x, t) = -2\nu s_x/s$, we obtain the viscous Burgers' equation above. This is the so-called Cole-Hopf transformation.
- Check that $s = 1 + a\exp(-kx + \nu k^2 t)$ is a solution of the heat equation. What solution of the viscous Burgers' equation does it correspond to ? (describe it qualitatively e.g. velocity, amplitude, spatial profile etc.) in terms of its parameters.

Practice Problems

1) Let's get a head start towards establishing moment identities as in problem 2) above. Try to establish the conservation of the mass in the KdV equation.

2) Solve the equation $u_t + uu_x = 0$ with initial condition $u(x, 0) = x^2$. Also solve the transport PDE $(1 + x^2)u_x + u_y = 0$ with $u(0, y) = y^3$.