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## **Nonlinear Waves in Granular Crystals**

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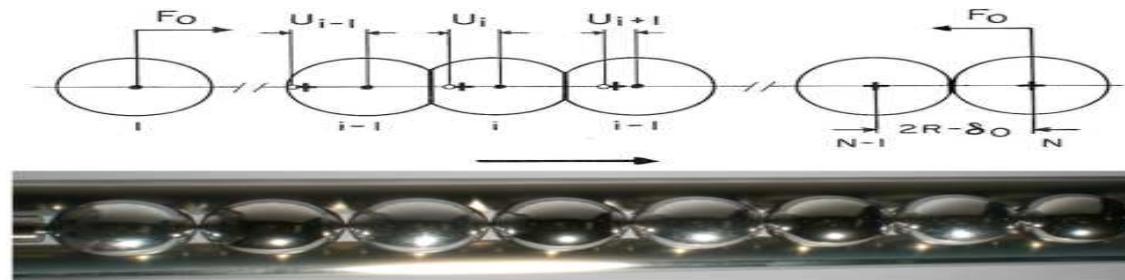
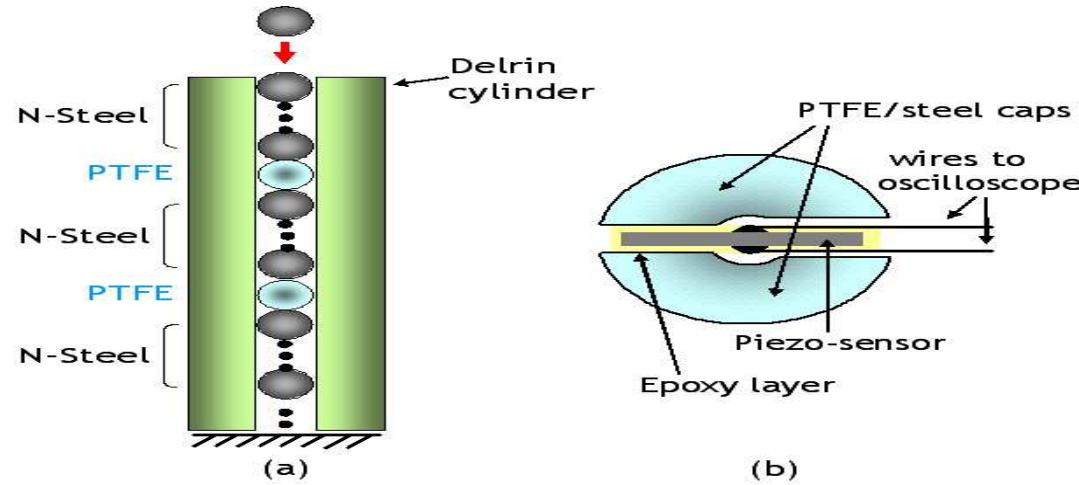
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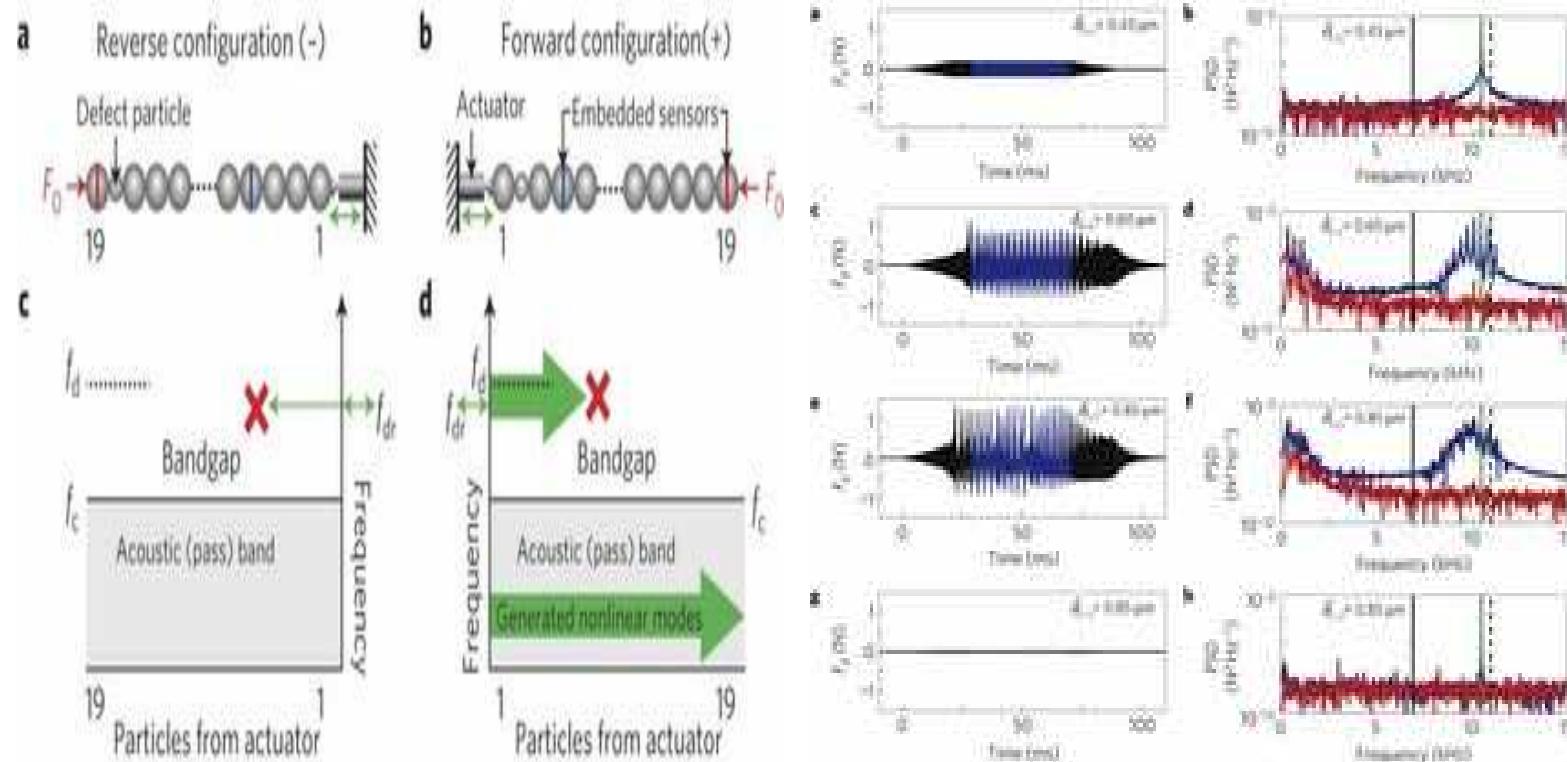
## Theoretical/Experimental Setup: Granular Crystals

### Experimental Configurations



## Very Simple but Very Useful !

- Use Granular Crystals to produce Acoustic Lenses and collect energy within Sound Bullets
- Use Granular Crystals to produce Acoustic Switching and induce Rectification



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## Theory/Numerics/Experiment Part 1: Traveling Waves

### Theoretical Setup

- Equations of Motion for Granular Crystal

$$\begin{aligned}\ddot{y}_j &= \frac{A_{j-1,j}}{m_j} (\delta_0 + y_{j-1} - y_j)_+^p - \frac{A_{j,j+1}}{m_j} (\delta_0 + y_j - y_{j+1})_+^p + g, \\ A_{j,j+1} &= \frac{4E_j E_{j+1} \left( \frac{R_j R_{j+1}}{R_j + R_{j+1}} \right)^{1/2}}{3 \left[ E_{j+1} (1 - \nu_j^2) + E_j (1 - \nu_{j+1}^2) \right]}. \end{aligned} \quad (1)$$

- For the Dimer with Time Rescaling and Without Gravity:

$$m_1 \ddot{u}_j = (w_j - u_j)_+^p - (u_j - w_{j-1})_+^p, \quad (2)$$

$$m_2 \ddot{w}_j = (u_{j+1} - w_j)_+^p - (w_j - u_j)_+^p, \quad (3)$$

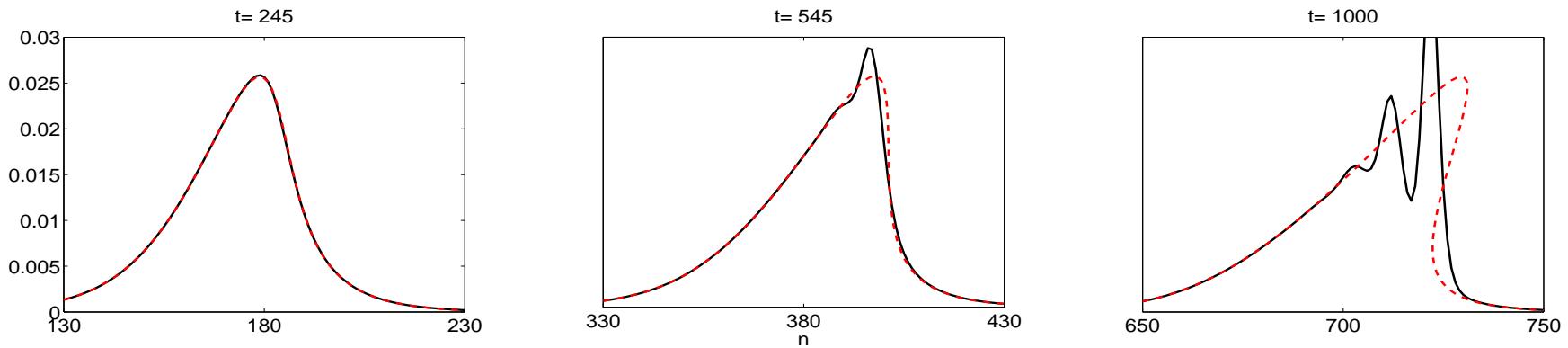
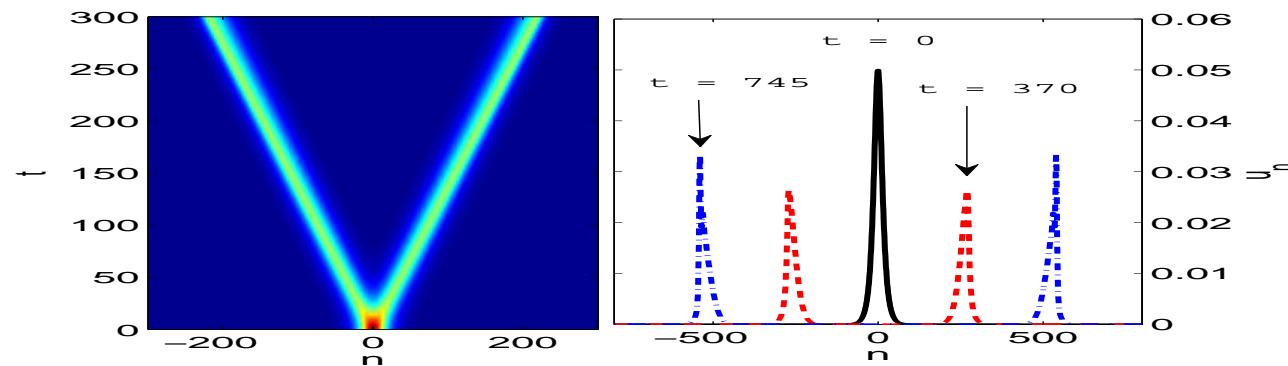
- $p = 3/2$ : Special Case of Hertzian Contacts.
- $m_1 = m_2$ : Case of Monomer Chain [with Precompression]

$$\ddot{u}_n = [\delta_0 + u_{n-1} - u_n]_+^p - [\delta_0 + u_n - u_{n+1}]_+^p. \quad (4)$$

## Typical Numerical Evolution (Part I)

- Use Strain Variables  $r_n = u_{n-1} - u_n$  to convert model to:

$$\ddot{r}_n = [r_{n+1}]_+^p - 2[r_n]_+^p + [r_{n-1}]_+^p, \quad (5)$$



## Theoretical Analysis of Monoatomic Crystal and Shock Waves

- Fix  $\delta_0 > 0$  and let  $p \in \mathbb{R}^+$  and let  $A \in C([0, T_0], H^4)$  with

$$\sup_{T \in [0, T_0]} \sup_{X \in \mathbb{R}} |A(X, T)| \leq \delta_0/2 \quad \text{and} \quad \sup_{T \in [0, T_0]} \|A(\cdot, T)\|_{H^4} \leq C_1$$

be a solution of the **Partial Differential Equation**

$$\partial_T^2 A = \partial_X^2 ((\delta_0 + A)^p). \tag{6}$$

Then for  $C_1 > 0$  sufficiently small there exists  $C, \varepsilon_0 > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  there are solutions  $(u_n(t))_{n \in \mathbb{Z}}$  of (5) satisfying

$$\sup_{t \in [0, T_0/\varepsilon]} \sup_{n \in \mathbb{Z}} |r_n(t) - A(\varepsilon n, \varepsilon t)| < C\varepsilon^{3/2}.$$

- One can also establish the approximations Let  $A \in C([0, T_0], H^6)$  be a solution of the **KdV equation**  $\partial_T A = \nu_1 \partial_X^3 A + \nu_2 \partial_X(A^2)$  with suitable chosen coefficients  $\nu_1, \nu_2 \in \mathbb{R}$ . Then there exist  $\varepsilon_0 > 0, C > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  we have solutions  $(u_n)_{n \in \mathbb{Z}}$  of the **discrete equation** with

$$\sup_{t \in [0, T_0/\varepsilon^3]} \sup_{n \in \mathbb{N}} |u_n(t) - \psi_n(t)| \leq C\varepsilon^{5/2},$$

where

$$\psi_n(t) = \varepsilon^2 A(\varepsilon(n - \omega'_1(0)t), \varepsilon^3 t)$$

with  $\omega_1(k) = \omega(k)^2 b_1$ .

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## Theoretical Analysis of Monoatomic Crystal and Shock Waves

- Let  $A \in C([0, T_0], H^{19})$  be a solution of the **NLS equation**  
 $\partial_T A = i\nu_1 \partial_X A + i\nu_2 A |A|^2$  with suitable chosen coefficients  $\nu_1, \nu_2 \in \mathbb{R}$ . Then there exist  $\varepsilon_0 > 0, C > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  we have solutions  $(u_n)_{n \in \mathbb{Z}}$  of the **discrete equation** with

$$\sup_{t \in [0, T_0/\varepsilon^2]} \sup_{n \in \mathbb{N}} |u_n(t) - \psi_n(t)| \leq C\varepsilon^{3/2}$$

where

$$\psi_n(t) = \varepsilon A (\varepsilon(n - \omega'_j(k_0)t), \varepsilon^2 t) e^{i(k_0 n - \omega_0 t)} + \text{c.c.}$$

with  $\omega_1(k) = \omega(k)^2 b_1$

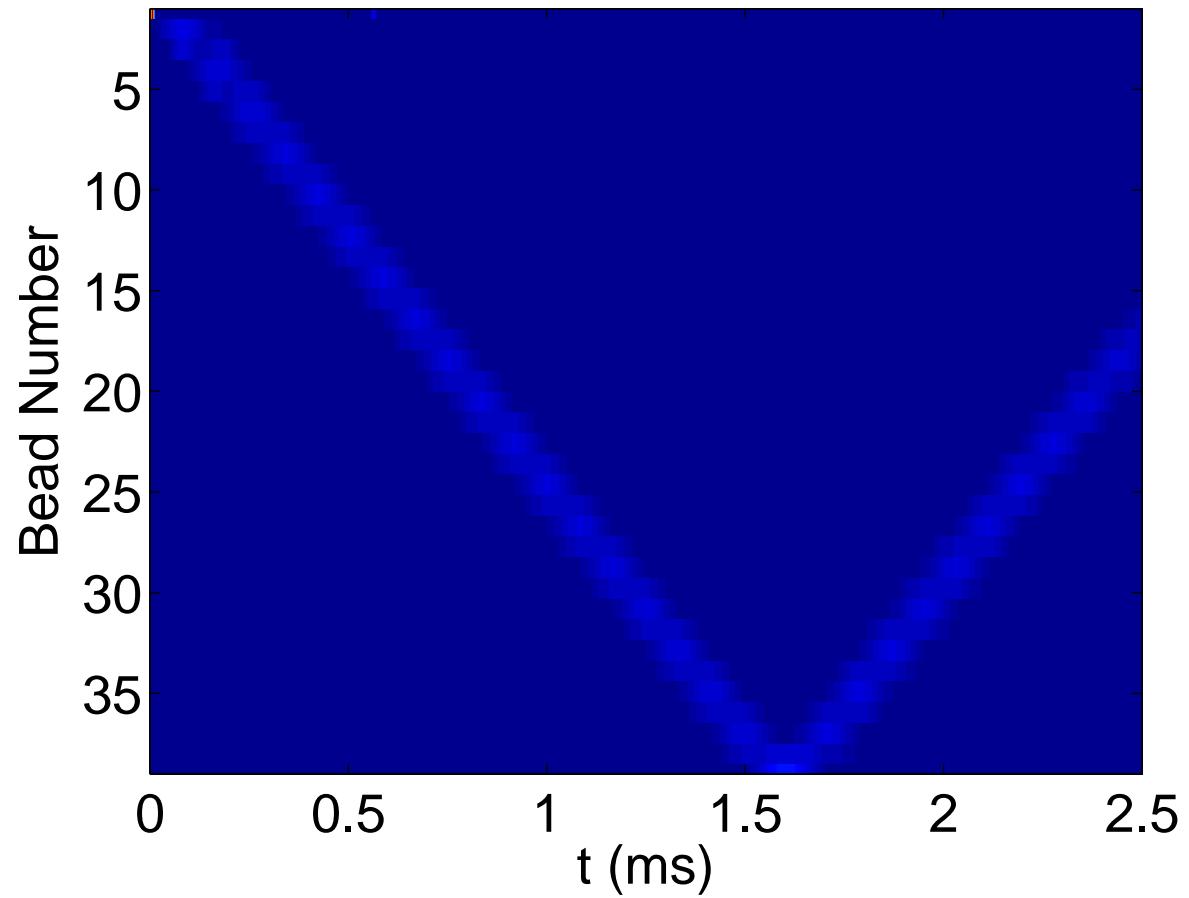
- This first of these **PDEs** leads to the **p-system**

$$\left. \begin{array}{l} \partial_T A - \partial_X v \\ \partial_T v - \partial_X [(\delta_0 + A)^p] \end{array} \right\} = 0 \quad (7)$$

- This leads to the **Eigenvalues**  $\lambda_{\pm}(U) = \pm\sqrt{p(\delta_0 + A)^{p-1}}$  and thus the propagation of **Counter-Propagating Waves** with Speed  $\pm\sqrt{p(\delta_0 + A)^{p-1}}$  and ultimately to **Wave Breaking** and formation of **Shock Waves**.

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## Typical Numerical Evolution (Part II)



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## Theoretical Analysis of Monoatomic Crystal

- Use Again **Strain Variables**  $r_n = u_{n-1} - u_n$  without or with precompression
- For **Traveling Waves**  $r_n = \phi(x) \equiv \phi(n - ct)$  obtain **Advance-Delay Equation**

$$c^2\phi''(x) = (\delta_0 + \phi^p(x-1)) + (\delta_0 + \phi)^p(x+1) - 2(\delta_0 + \phi)^p(x) \quad (8)$$

- For  $c = 1$ , Apply **Fourier transform** to obtain  $\hat{\phi}(k) = \left(\frac{4 \sin^2(\frac{k}{2})}{k^2}\right) (\delta_0 + \hat{\phi})^p(k)$
- And **Convolution Theorem** to get **Algorithmic Fixed Point Scheme**:

$$\phi(x) = \frac{(\Lambda \star (\delta_0 + \phi)^p)(x)}{\int (\Lambda \star \phi^p)(x) dx}. \quad (9)$$

where  $\Lambda(x) = (1 - |x|)_+$ .

- Upon convergence of scheme to **Nontrivial Profile**, for  $\delta_0 = 0$ , notice that:

$$\phi(x+1) = \int_{-1}^1 \Lambda(y) \phi^p(x+1-y) dy \leq \phi^p(x) \Rightarrow \phi(x+n) \leq \phi(x)^{p^n}, \quad (10)$$

- In the presence of **Precompression**, we obtain:

$$\phi(x+1) \sim \delta_0^{p-1} \phi(x) \Rightarrow \phi(x) \sim \delta_0^{p-1} \exp(n \log r(x_0)) \quad (11)$$

## A Side Note: Proof of Bell-Shaped Solutions of Monoatomic Crystal

- Consider the **Operator**:  $Qf(x) = \int_{x-1/2}^{x+1/2} f(y)dy$ , for which it can be seen that:  $\widehat{Qf}(\xi) = \frac{\sin(\pi\xi)}{\pi\xi} \widehat{f}(\xi)$  and that the **Convolution Operator**  $M$  considered previously is simply  $M = Q^2$ .
- Set up the **Constrained Optimization Problem** with  $q = 1 + \frac{1}{p} \in (1, 2)$

$$\begin{cases} J_\varepsilon(v) = \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} |Qv(x)|^2 dx \rightarrow \max \\ \text{subject to } I(v) = \int v^q(x) dx = 1, \\ v \geq 0, v \text{ bell-shaped} \end{cases} \quad (12)$$

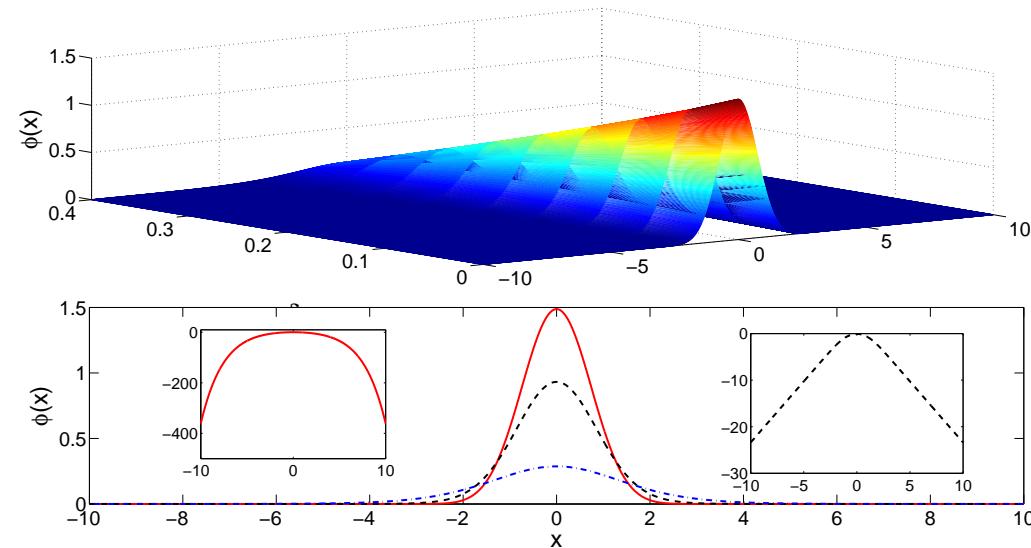
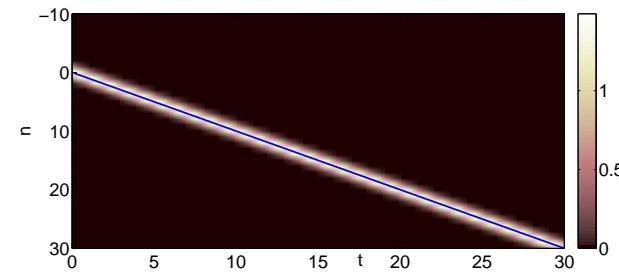
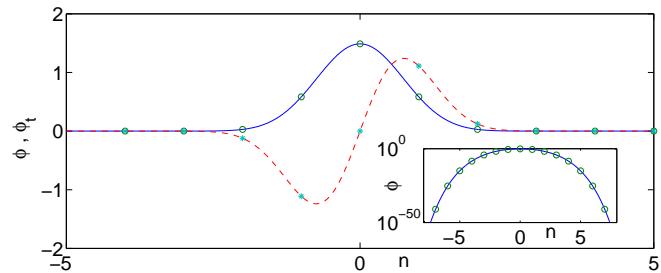
- It can be proved that this problem **posseses a solution** which corresponds to the **bell-shaped traveling wave** solution of the monoatomic problem in the strain variables.
- The **Resulting Solution** satisfies the **Euler-Lagrange equation** (equivalent to the **Fixed Point Formulation**)

$$Mv - J_\varepsilon^{\max} v^{q-1} = 0.$$

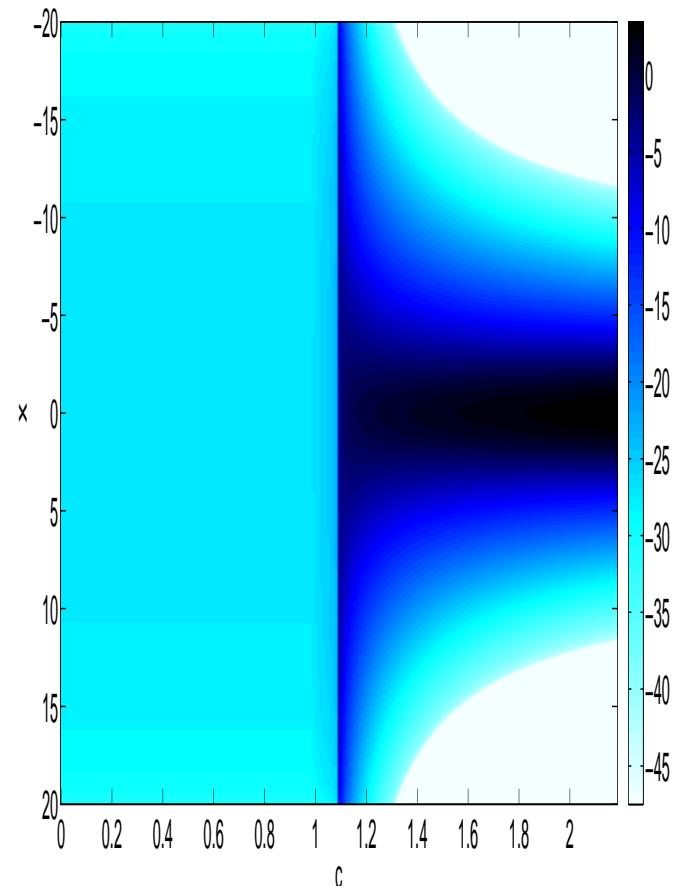
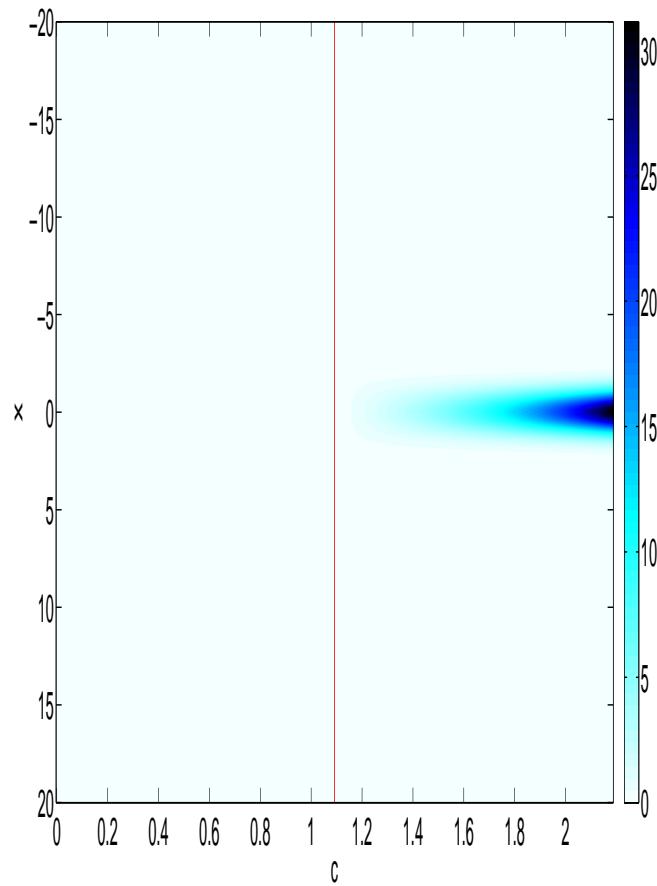
- Interestingly, the well-known **Friesecke-Wattis result** for **traveling waves in lattices** becomes a **special case** where bell-shapedness is not required.
- The **existence of TWs** can be extended to **finite  $\delta_0$** . There, TWs will exist only for  $c > c_s = \sqrt{p} \delta_0^{(p-1)/2}$ .

## Numerical Results for Monoatomic Crystal

### The Route from (Near-)Compactons to Solitons



## Traveling Waves with Precompression (only above the Sound Speed)



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## Another Approach to the Monoatomic Crystal

### Less Rigorous but More Explicit

- Use a Continuum Long Wavelength Approximation  
$$u_{n\pm 1} = u \pm hu_x + (h^2/2)u_{xx} \pm (h^3/6)u_{xxx} + (h^4/24)u_{4x}$$
- Obtain the following Nesterenko equation for Displacement & Strain:

$$u_{tt} = (u_x^p)_x + \frac{h^2}{12} \left( (u_x^p)_{xxx} - \frac{n(n-1)}{2} (u_x^{p-2} u_{xx}^2)_x \right) \quad (13)$$

$$r_{tt} = (r^p)_{xx} + \frac{h^2}{12} \left( (r^p)_{xxxx} - \frac{n(n-1)}{2} (r^{p-2} r_x^2)_{xx} \right) \quad (14)$$

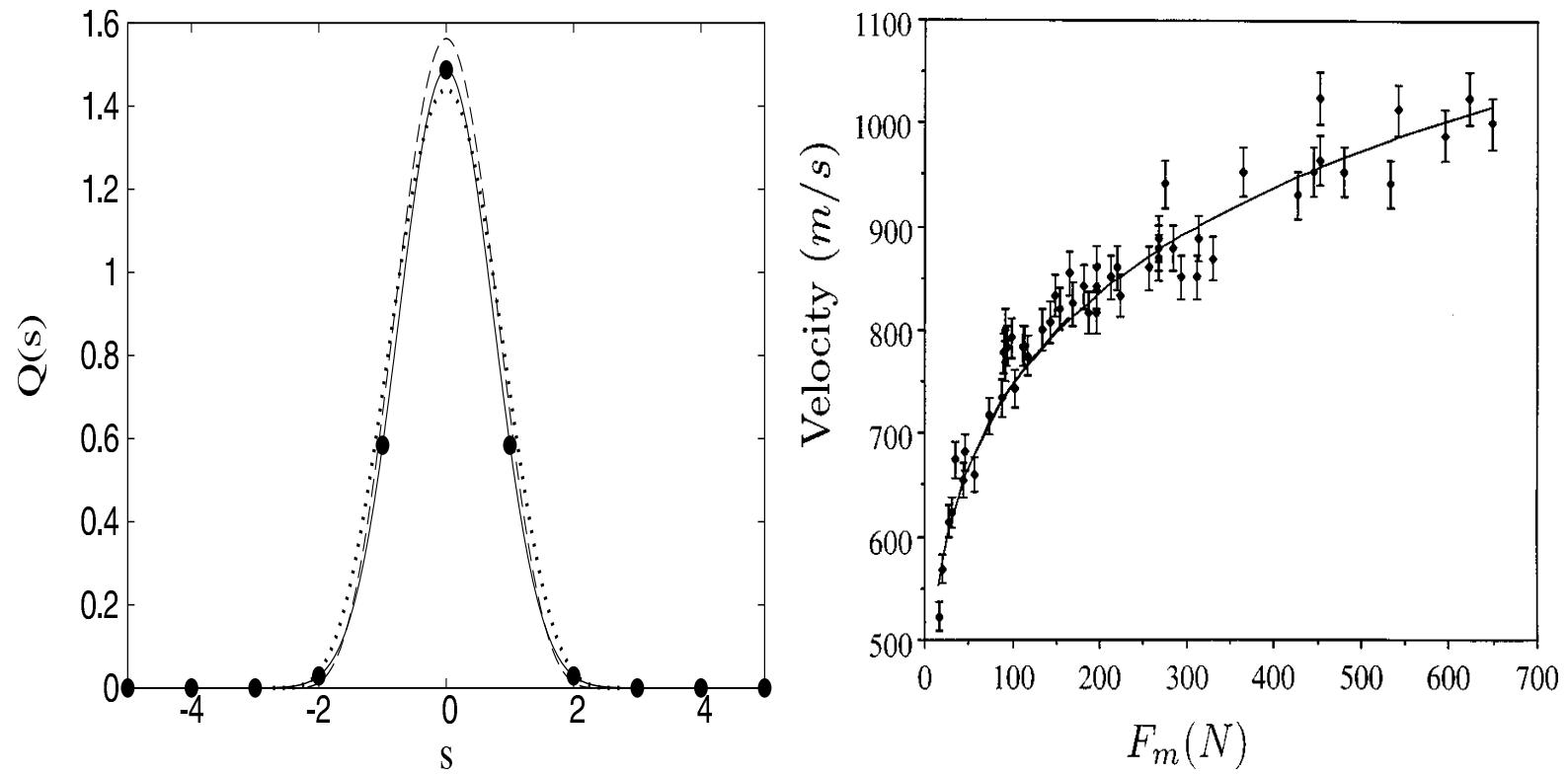
- This leads to the Explicit Compacton Solution (when truncated)  
 $r(x - ct) = c^m A \cos^m(B(x - ct))$  with  $m = 2/(p - 1)$  and  
 $B = \sqrt{6(p-1)^2/(p(p+1))}$ . [Note:  $B = \sqrt{2/5}$  for Hertzian  $p = 3/2$ ].
- Note that if one uses Long Wavelength Approximation directly in Strain Equation (Ahnert-Pikovsky):

$$r_{tt} = (r^p)_{xx} + \frac{h^2}{12} (r^p)_{xxxx}. \quad (15)$$

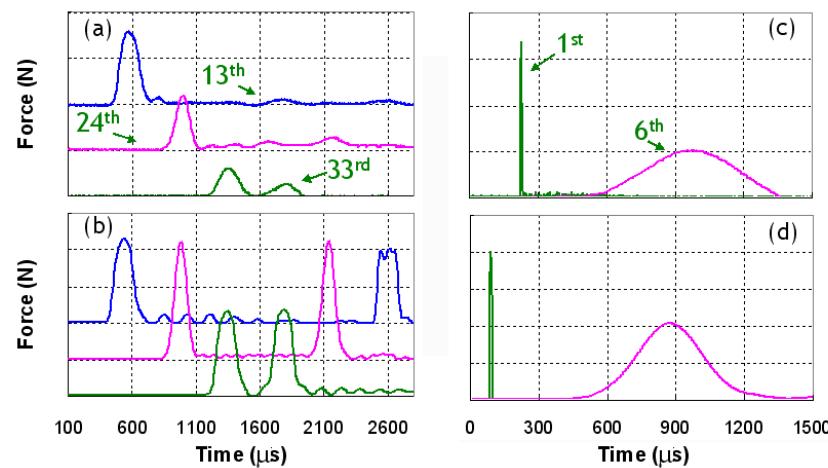
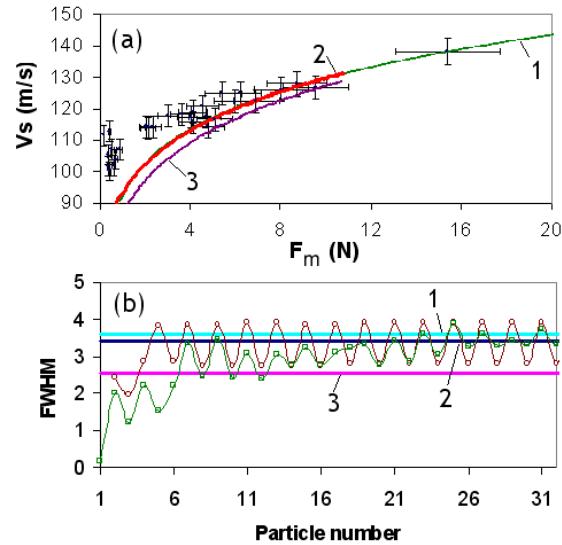
- This leads to the Explicit Compacton Solution (when truncated)  
 $r(x - ct) = c^m A \cos^m(\tilde{B}(x - ct))$  with  $m = 2/(p - 1)$  and  $\tilde{B} = \sqrt{3}(p-1)/p$ .

## Comparison of Long-Wavelength Approximation with Numerics/Experiments

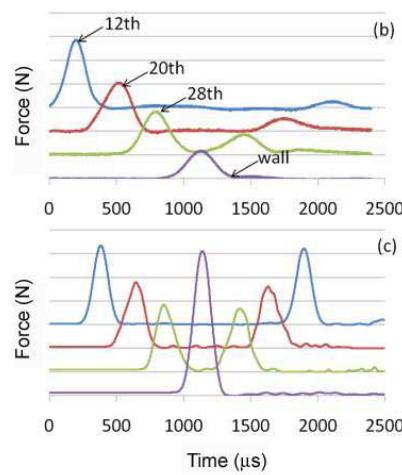
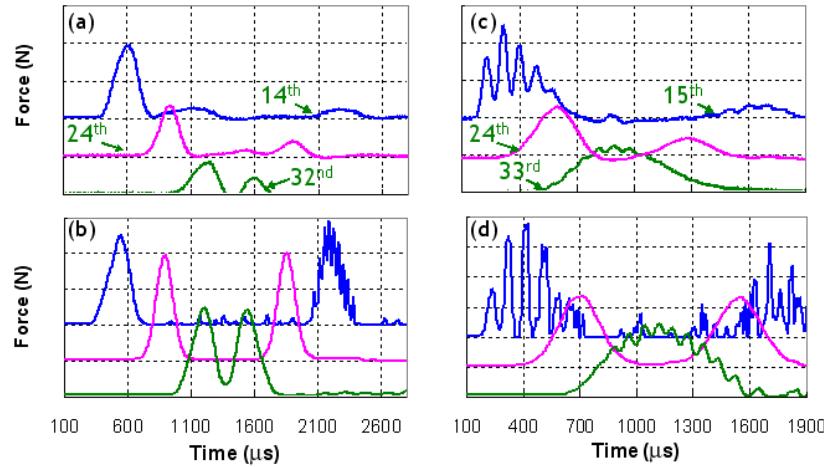
Monoatomic Case (Ahnert-Pikovsky, Coste-Falcon-Fauve)



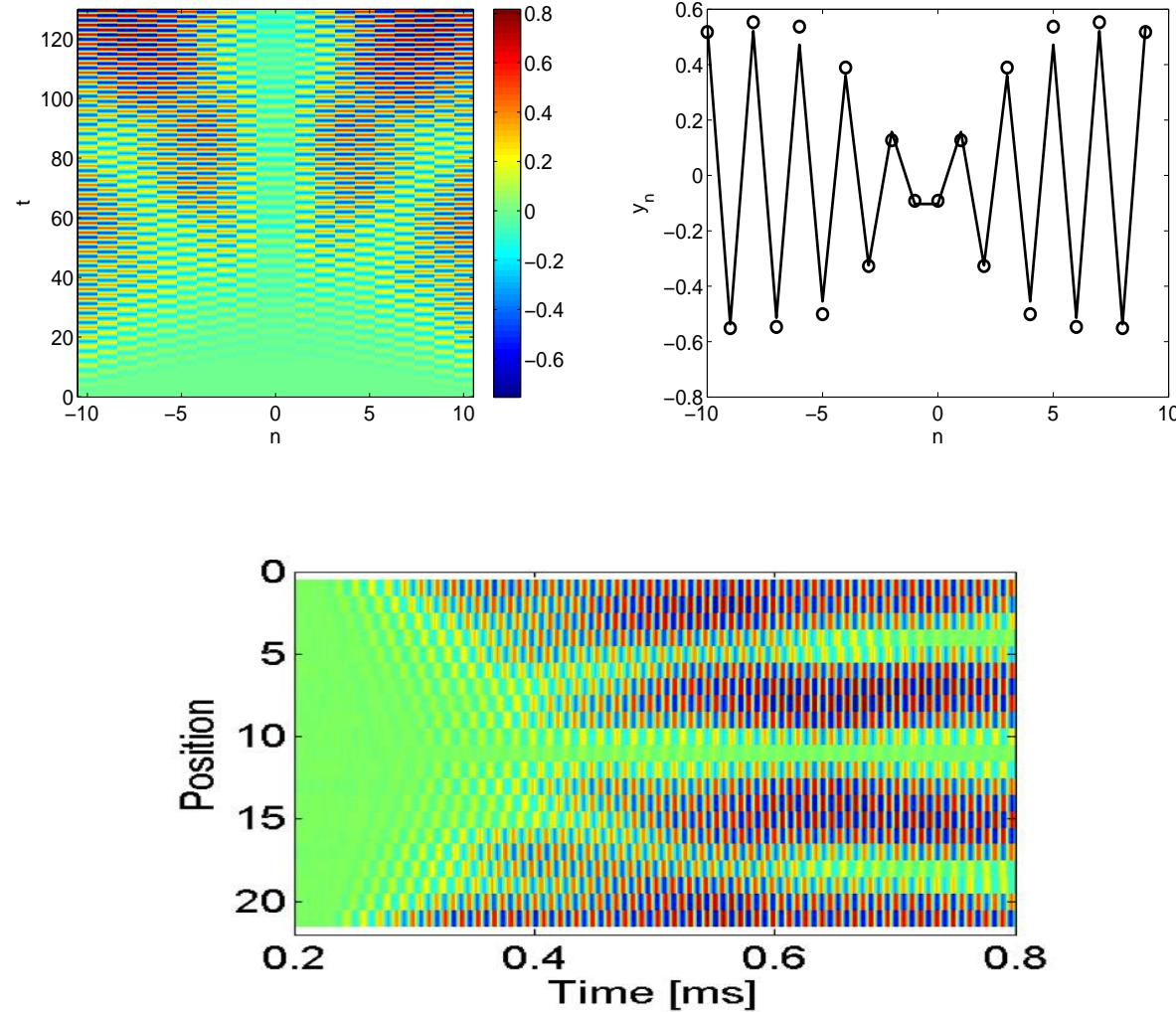
## Generalization to Higher Numbers of Beads: Dimer of 1:1 Beads



## More Complex Chains: 2:1, 5:1 and 1:1:1 Trimers



## (A)typical Numerical Evolution (Part III)



## Dark Breathers in the Monoatomic Chain

- In the **Strain Variables**  $r_n$  the **Hertzian contact** model with  $V'_{\text{GC}}(x) = A [\delta_0 - x]_+^{3/2}$  ( $A, \delta_0$  constants) can be **FPU-approximated** as

$$M\ddot{r}_n = V'_{\text{FPU}}(r_{n+1}) - 2V'_{\text{FPU}}(r_n) + V'_{\text{FPU}}(r_{n-1}). \quad (16)$$

with  $V'_{\text{FPU}}(x) = K_2x + K_3x^2 + K_4x^3$  and

$$K_2 = \frac{3}{2}A\delta_0^{1/2}, \quad K_3 = -A\frac{3}{8}\delta_0^{-1/2}, \quad K_4 = -A\frac{3}{48}\delta_0^{-3/2}.$$

- Using a **Mutiple Scale Ansatz**  $y_n(t) \approx \psi_n(t) := \epsilon A(X, T)e^{i(k_0 n + \omega_0 t)}$  + c.c., with  $X = \epsilon(n + ct)$ ,  $T = \epsilon^2 t$ , a **Nonlinear Schrödinger Model** can be derived:

$$i\partial_T A(X, T) + \nu_2 \partial_X^2 A(X, T) + \nu_3 A(X, T)|A(X, T)|^2 = 0, \quad (17)$$

where  $\nu_2 = -\omega''(k_0)/2 > 0$  and  $\nu_3 = \frac{K_3^2}{K_2^2}\tilde{\gamma} + \frac{3K_4}{2K_2}\omega(k_0)$ , while

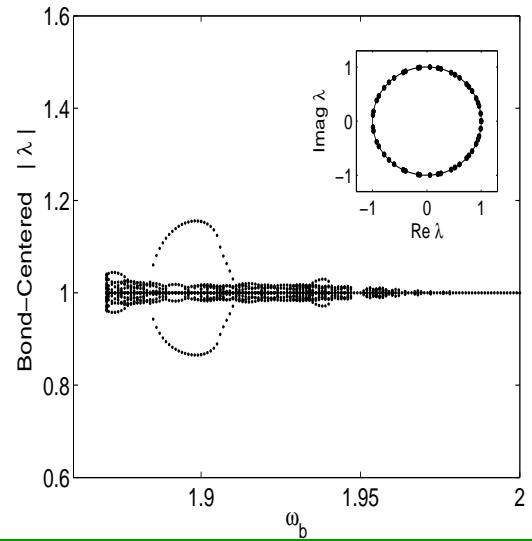
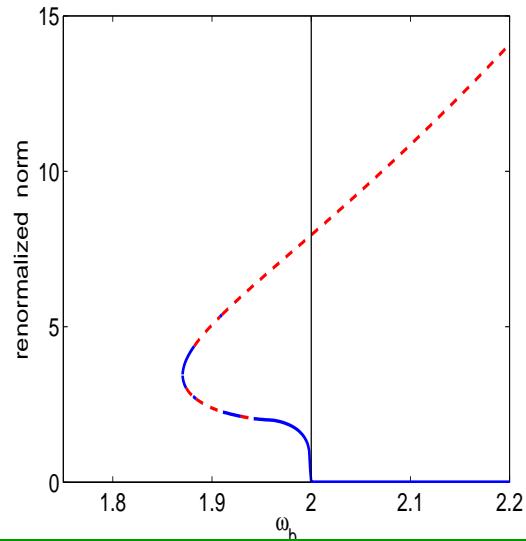
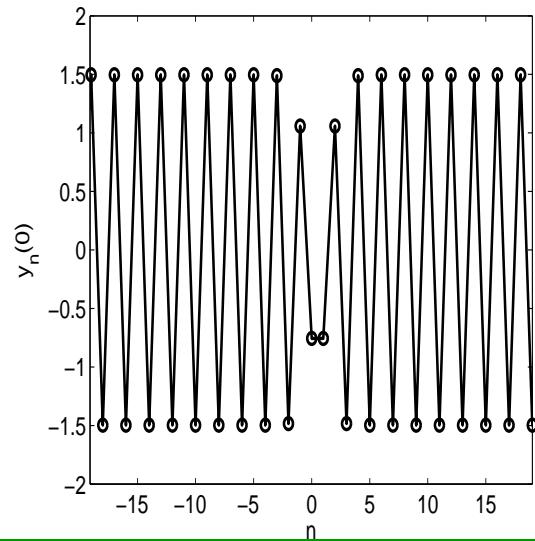
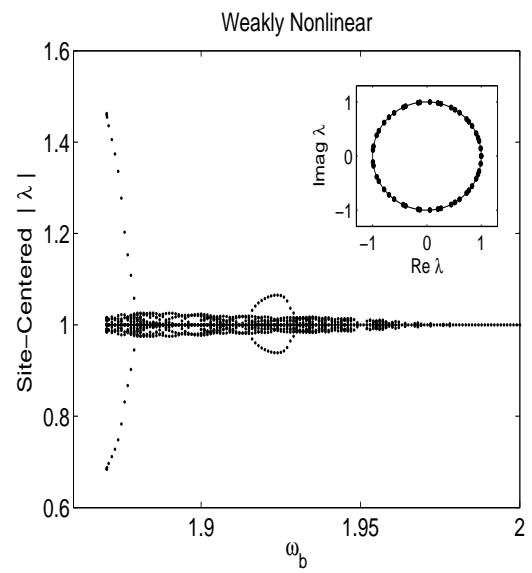
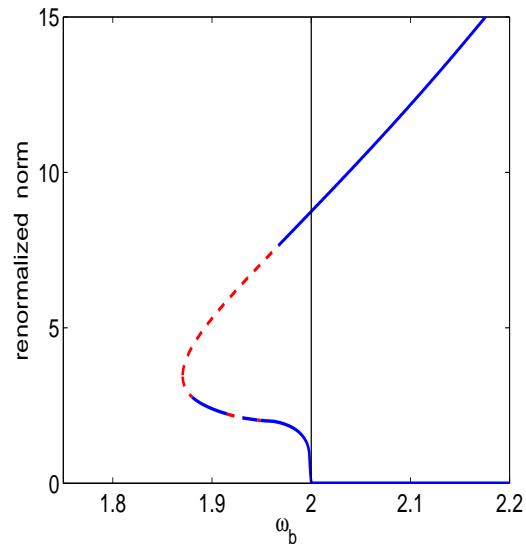
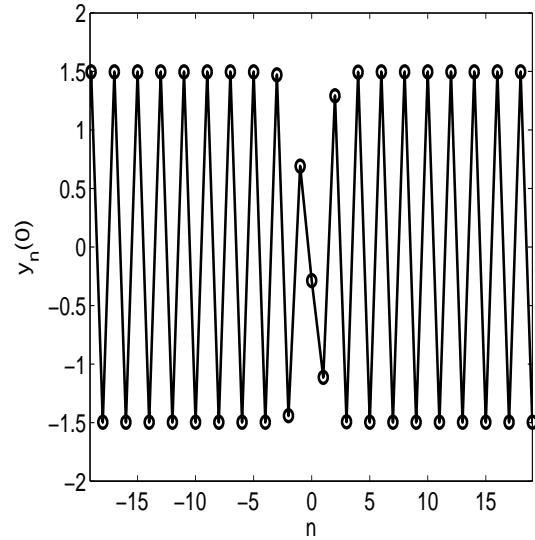
$$\tilde{\gamma} = \frac{\omega(k_0)}{2} \left( \frac{\omega(2k_0)}{2\omega(k_0) - \omega(2k_0)} - \frac{\omega(2k_0)}{2\omega(k_0) + \omega(2k_0)} + \frac{2\omega'(0)}{\omega'(k_0) - \omega'(0)} - \frac{2\omega'(0)}{\omega'(k_0) + \omega'(0)} \right).$$

- Seeking **Standing Wave Solutions**, we look close to  $k_0 = \pi$  and  $\omega_0 = 2\sqrt{K_2/M}$ , thus  $\nu_3|_{k_0=\pi} = 3K_2K_4 - 4K_3^2 = B < 0$ .
- The relevant explicit solution is the **Dark Breather**

$$y_n(t) = 2\epsilon(-1)^n \sqrt{\frac{\kappa}{\nu_3}} \tanh \left( \sqrt{\frac{-\kappa}{2\nu_2}} \epsilon(n - x_0) \right) \cos(\omega_b t) \quad (\text{for } k_0 = \pi)$$

- Solution can be created spontaneously by **Out of Phase Actuation of the Boundaries** in **Excellent Agreement** with Analytical Expression.

## Numerical Results for Dark Breathers



## More Complex Excitations: Defect Modes and Bright Breathers

### No Bright Breathers in Monoatomic Lattices & Role of Defects

- The Monoatomic Case  $M\ddot{u}_i = A[\delta_0 + u_{i-1} - u_i]_+^{3/2} - A[\delta_0 + u_i - u_{i+1}]_+^{3/2}$ , for  $\frac{|u_{i-1} - u_i|}{\delta_0} \ll 1$  can be expanded as:

$$M\ddot{u}_i = \sum_{j=2}^4 K_j ((u_{i+1} - u_i)^{j-1} - (u_{i-1} - u_i)^{j-1}) ; K_2 = \frac{3}{2} A \delta_0^{1/2} K_3 = -\frac{3}{8} A \delta_0^{-1/2}$$

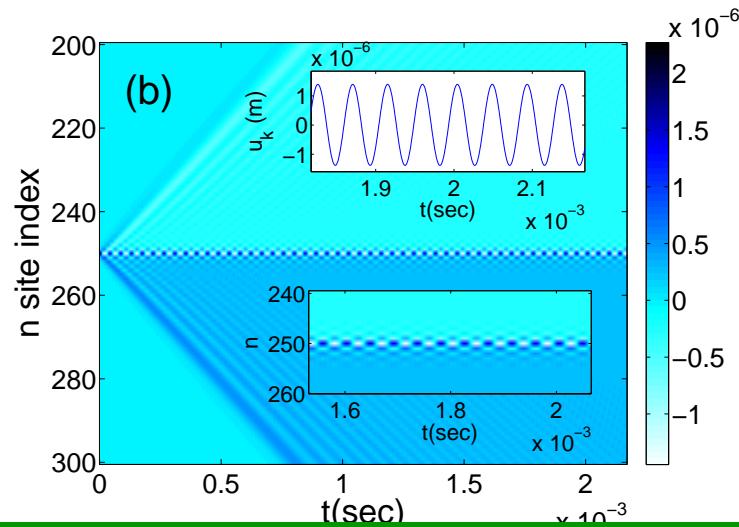
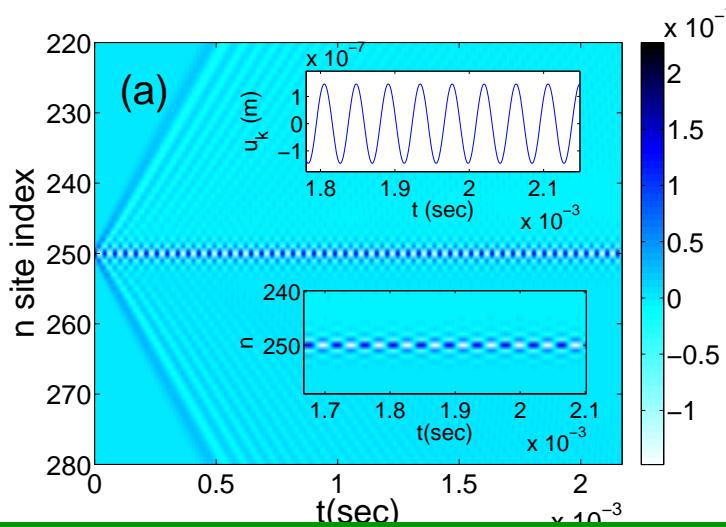
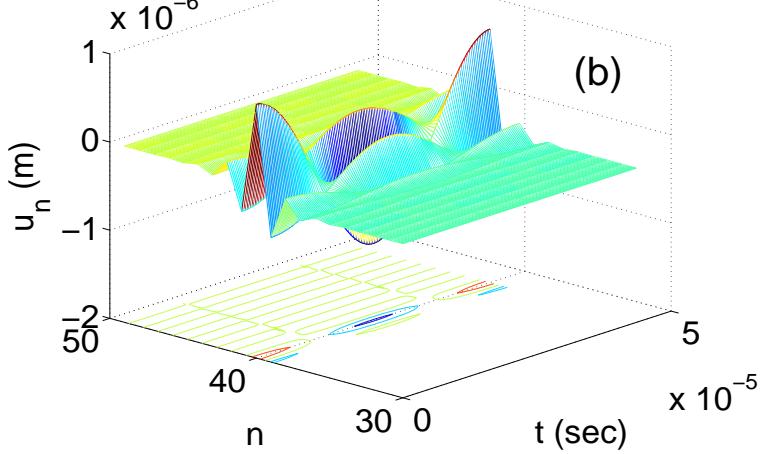
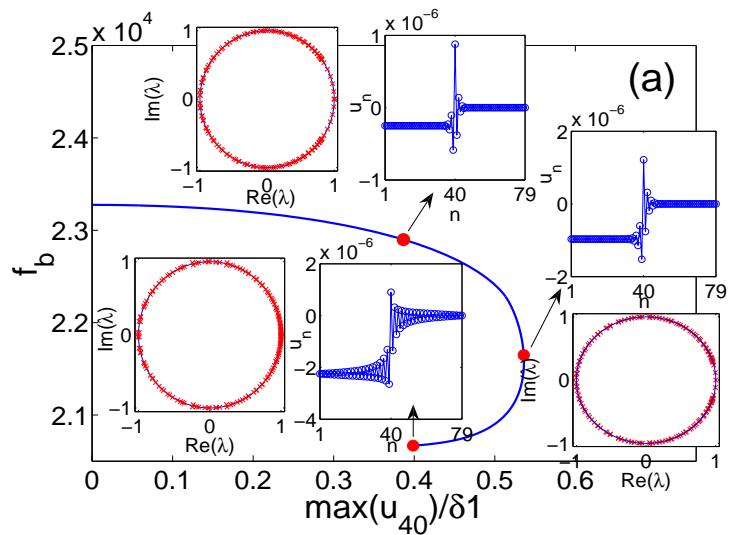
- Bifurcation Theory for FPU Problems indicates that from the Cutoff Frequency  $\omega_M = \sqrt{4K_2/M}$ , there exists a bright Breather Bifurcation if  $3K_2K_4 - 4K_3^2 > 0$  NOT true here !
- One Way Out: Use Defect(s) to Seed Breathing Impurity Modes

$$\begin{aligned} M\ddot{u}_{k-1} &= A_1[\delta_0 + u_{k-2} - u_{k-1}]_+^{3/2} - A_2[\delta_1 + u_{k-1} - u_k]_+^{3/2}, \\ m\ddot{u}_k &= A_2[\delta_1 + u_{k-1} - u_k]_+^{3/2} - A_2[\delta_1 + u_k - u_{k+1}]_+^{3/2}, \\ M\ddot{u}_{k+1} &= A_2[\delta_1 + u_k - u_{k+1}]_+^{3/2} - A_1[\delta_0 + u_{k+1} - u_{k+2}]_+^{3/2}, \\ A_1 &= \frac{2E \left(\frac{R}{2}\right)^{1/2}}{3(1-\nu^2)}, \quad A_2 = \frac{2E \left(\frac{Rr}{R+r}\right)^{1/2}}{3(1-\nu^2)}, \end{aligned} \tag{18}$$

- This model possesses Linear Spectrum Frequency  $f_a \approx \frac{\sqrt{3}}{2\pi} \frac{A_2^{1/3} F_0^{1/6}}{m^{1/2}}$

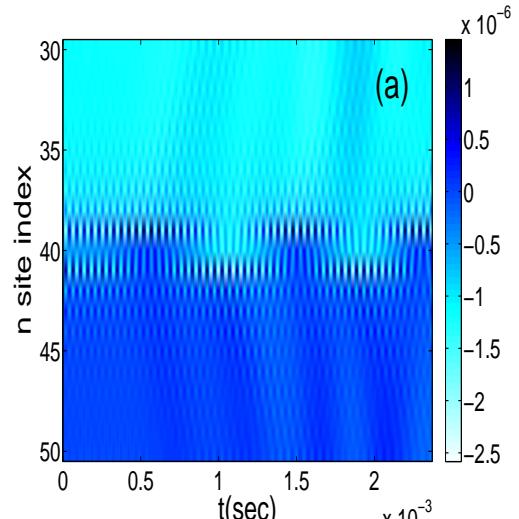
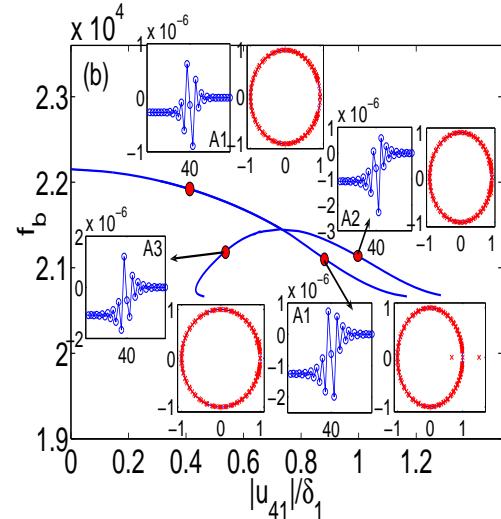
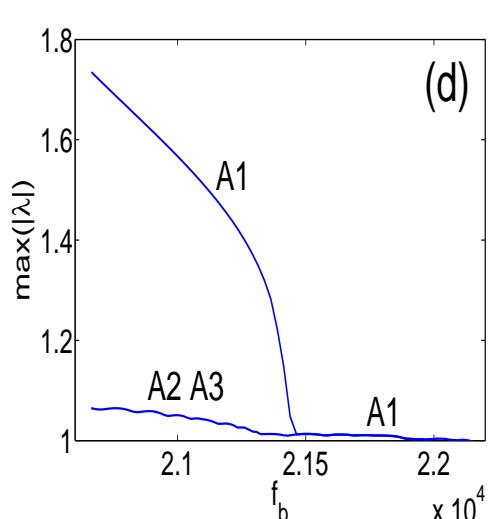
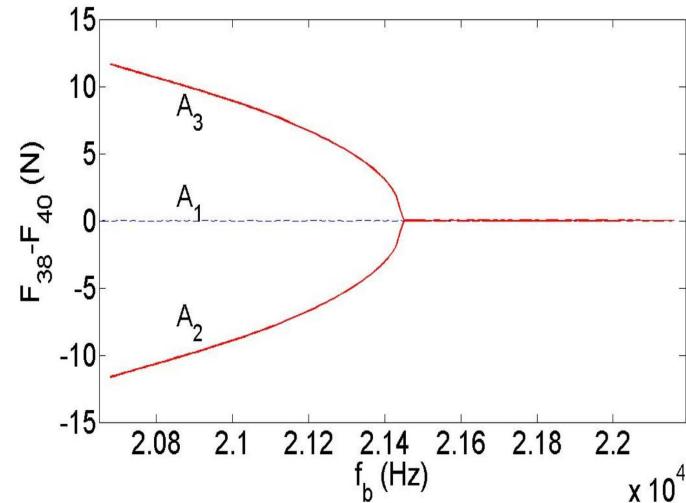
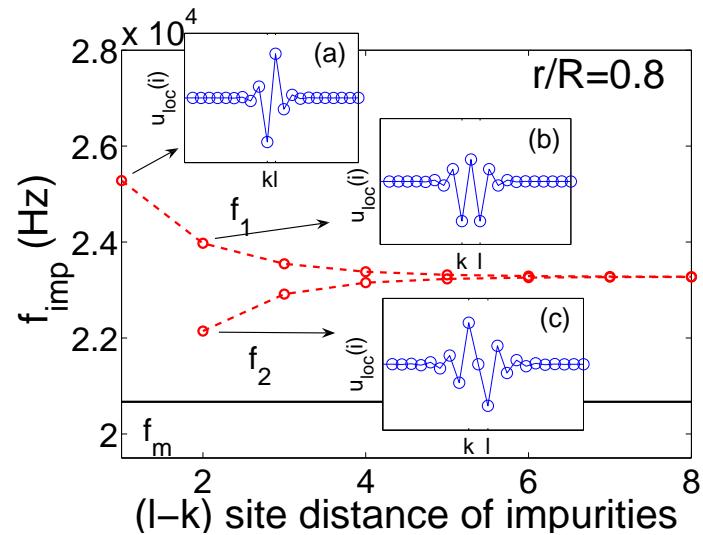
# The Role of Defects: Localized Nonlinear Breathing Modes

## Case of One Defect



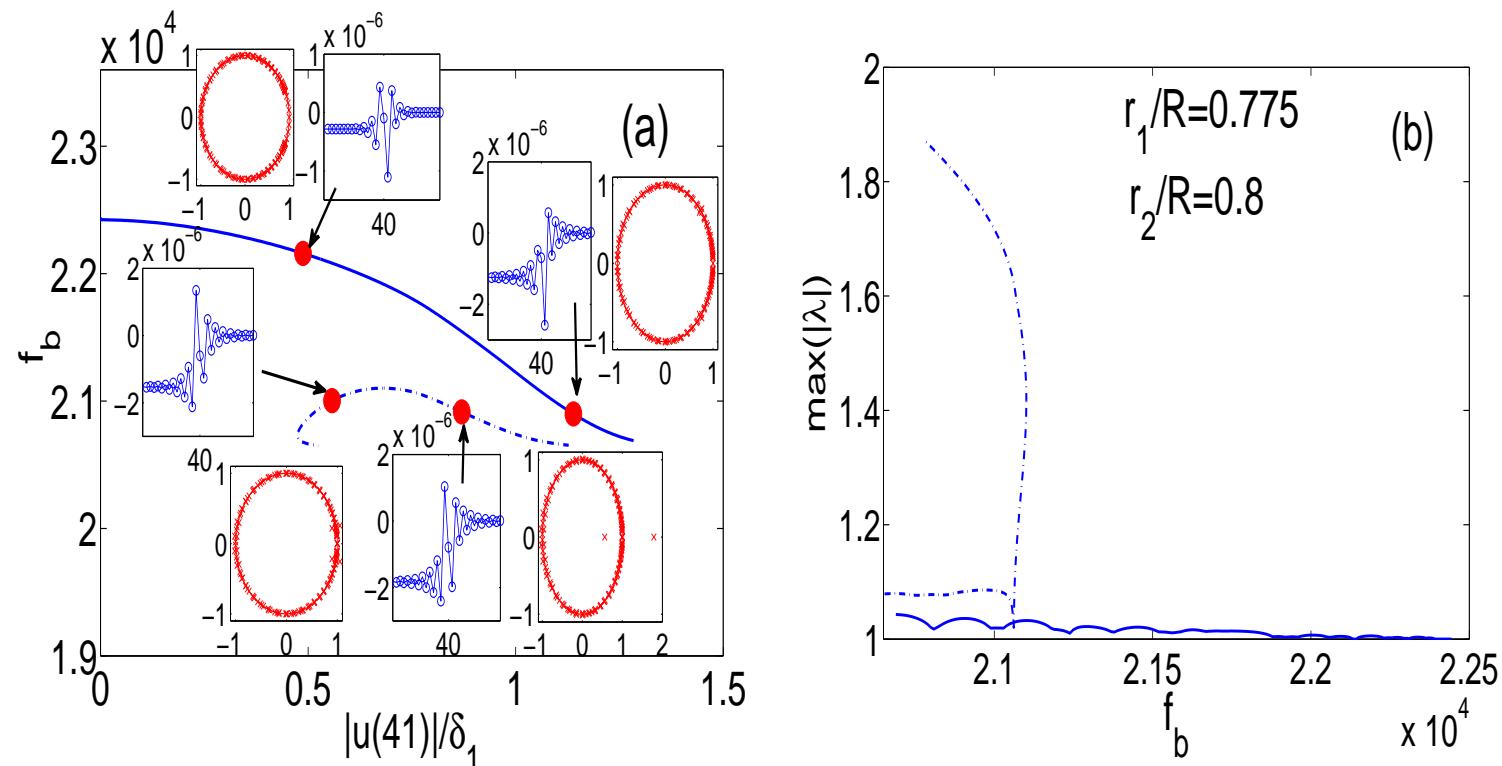
# The Role of Defects: Localized Nonlinear Breathing Modes

## Case of Multiple (Actually, Two !) Defects



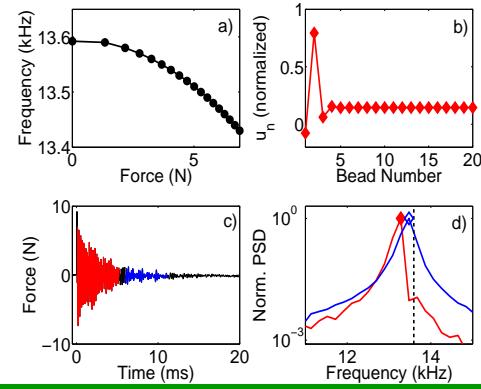
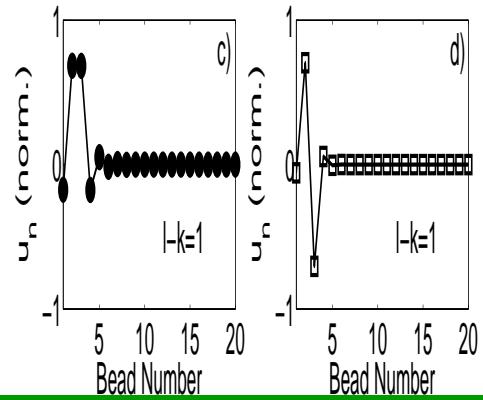
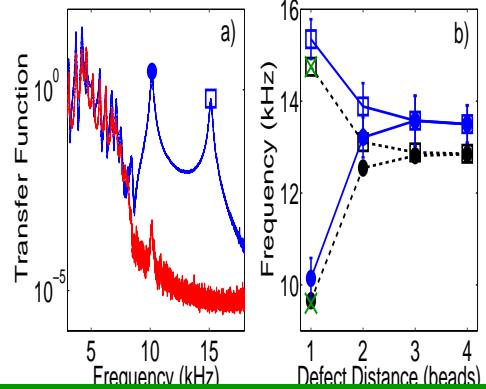
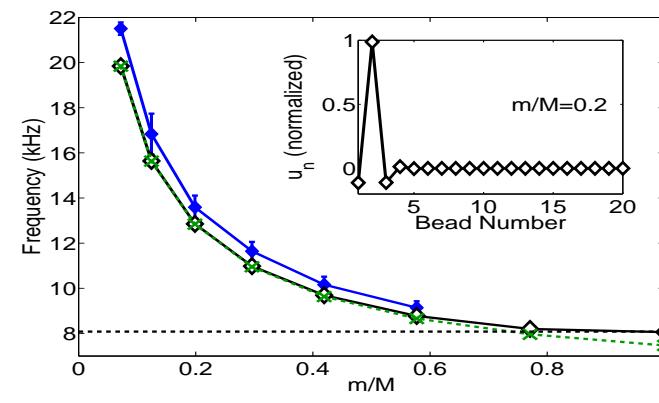
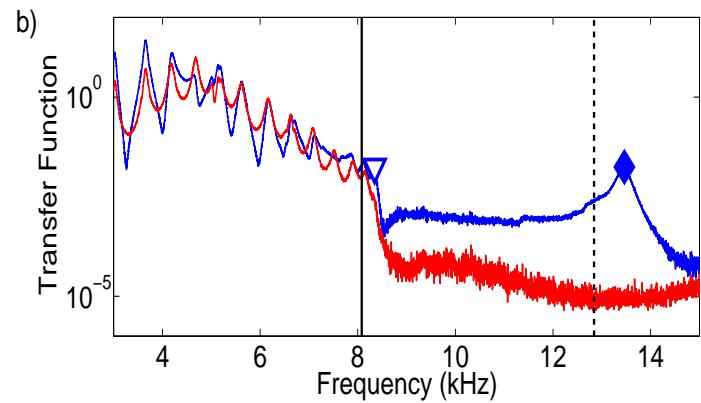
## The Role of Defects

### Imperfect Pitchfork Bifurcation from Asymmetric Defects



## First Set of Experimental Results on Defects

$$f_{3bead} = \frac{1}{2\pi} \sqrt{\frac{2K_{Rr}M + K_{RR}m + K_{Rr}m + \sqrt{-8K_{Rr}K_{RR}mM + [2K_{Rr}M + (K_{RR} + K_{Rr})m]^2}}{2mM}}$$

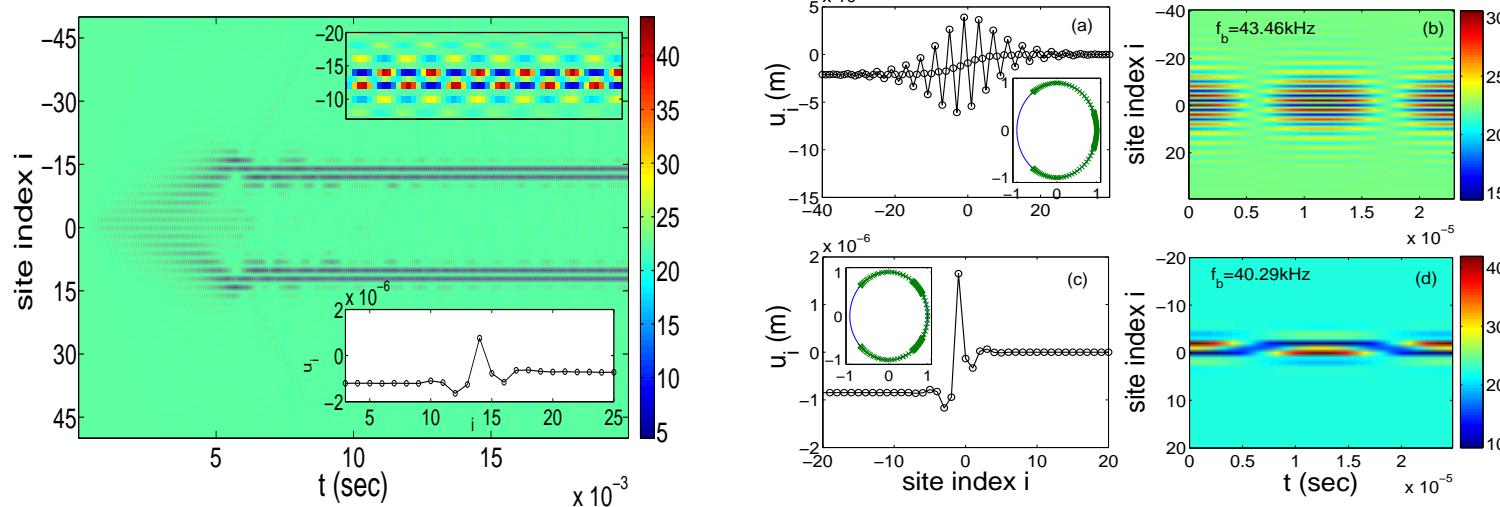


## The 1, 2, 3, ...∞ Program: The Dimer Crystal

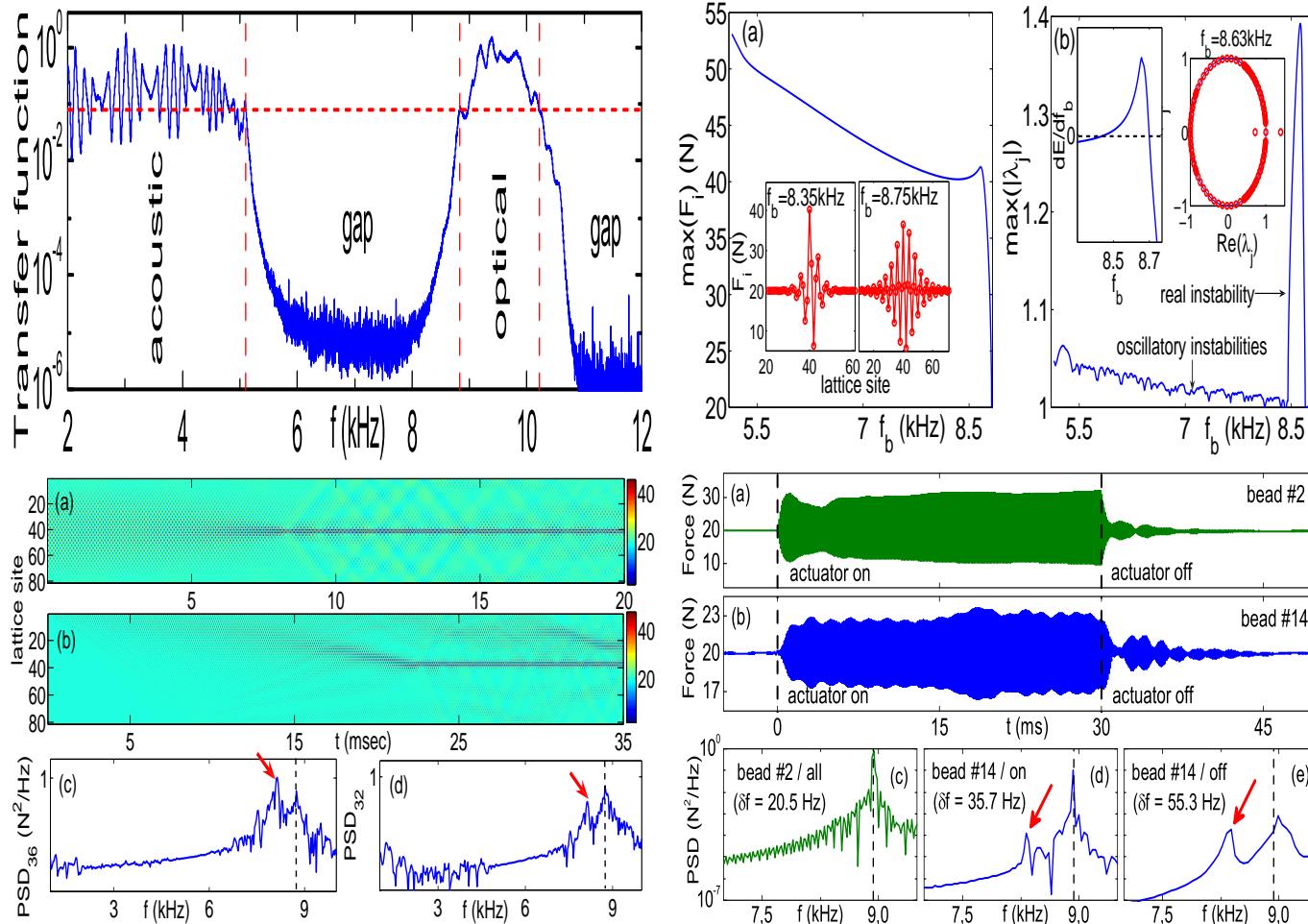
- Explicit Dispersion Relation:

$$\omega = \frac{K_2}{mM} \left[ m + M + (-1)^j \sqrt{m^2 + M^2 + 2mM \cos(2\alpha k)} \right] \quad (19)$$

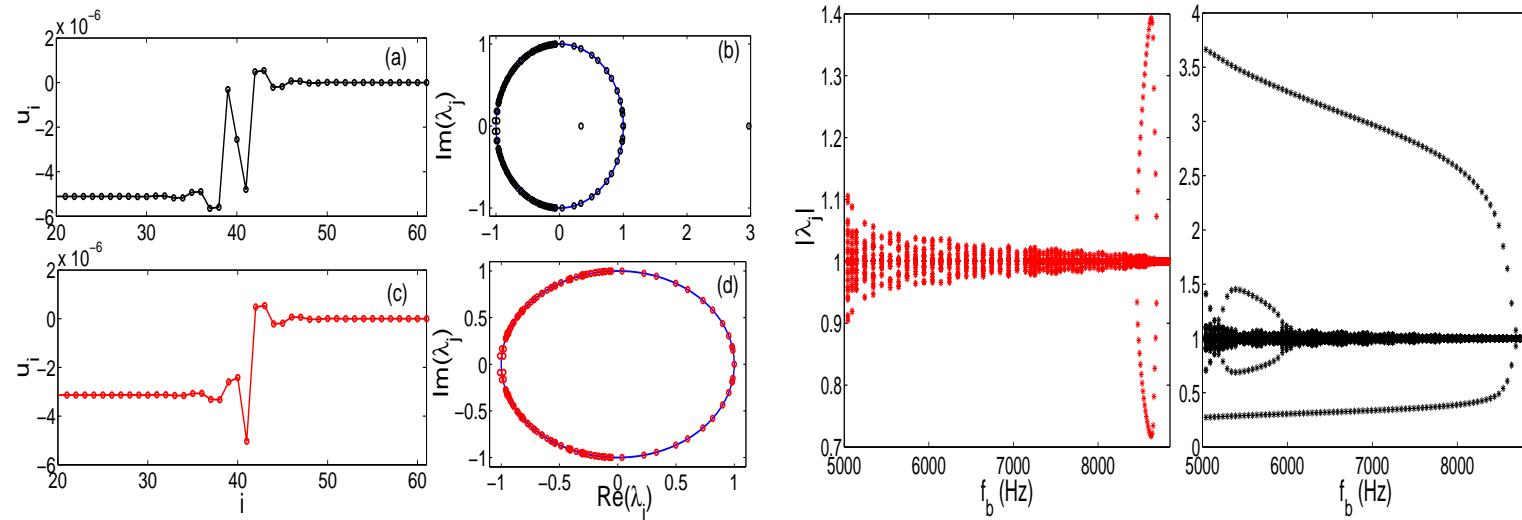
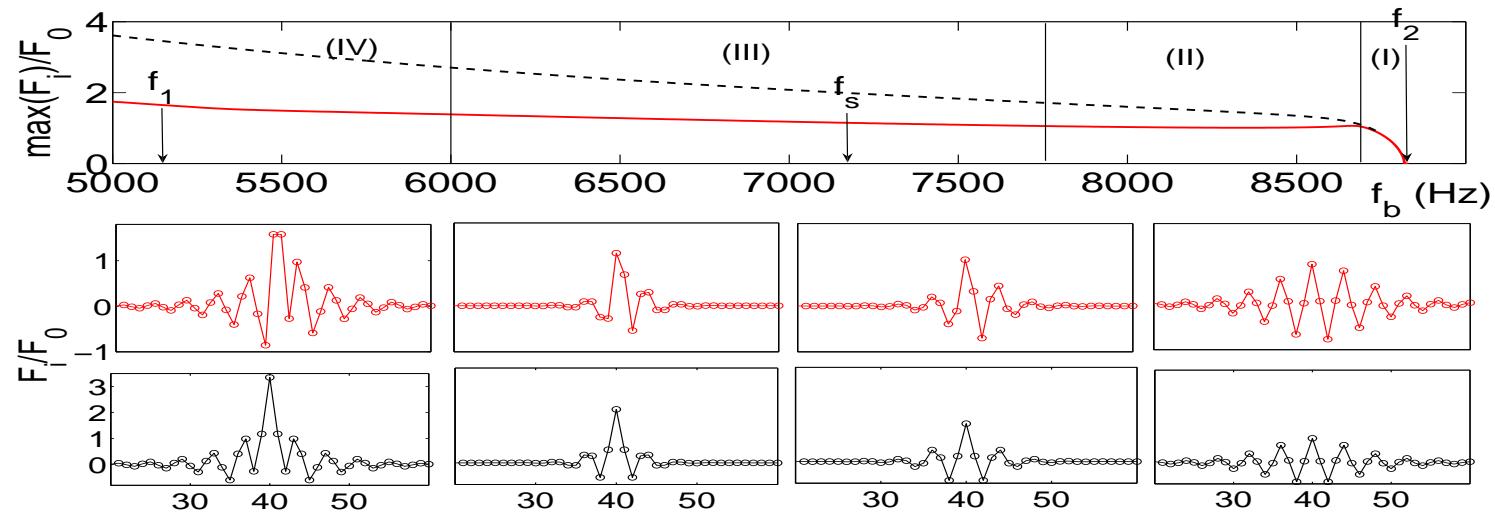
- Similar Condition  $3K_2K_4 - 4K_3^2 < 0$  leads to Modulational Instability (predicted by Huang) of Lower Optical Branch Cutoff
- This, in turn, results in Bright Discrete Breather Formation



## Experimental Observation of Discrete Breathers

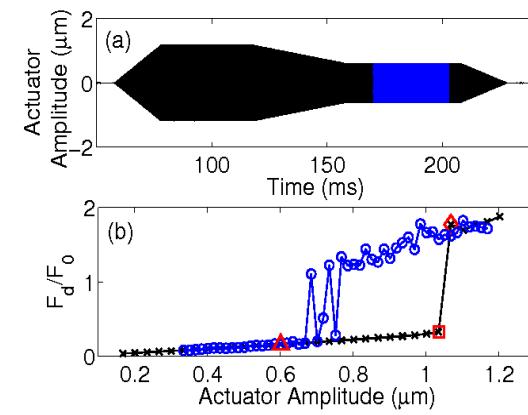
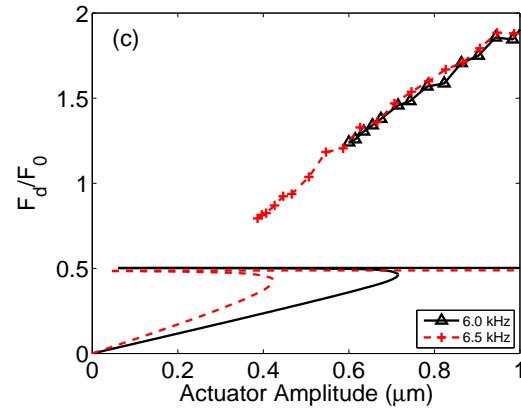
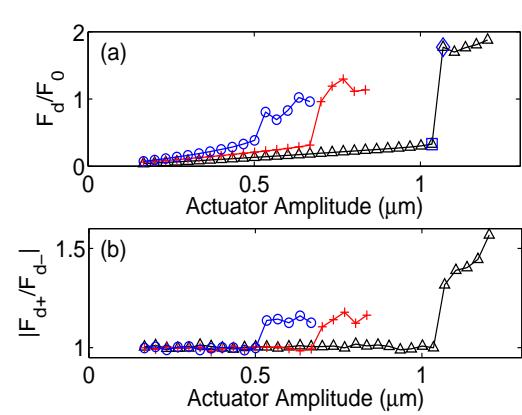
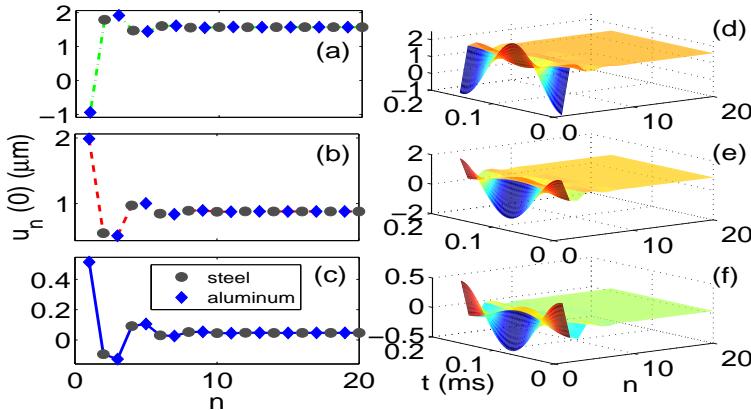
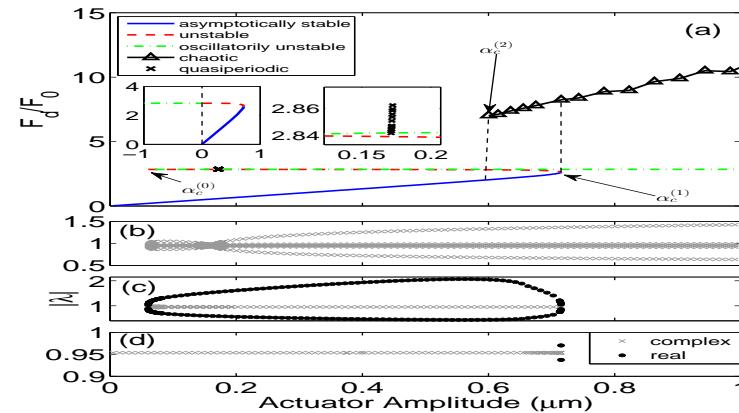


## A Systematic Exploration of Discrete Bulk Breathers



## Now Consider Periodically Driven/Damped Variant of the Model

### Hysteresis Loops, Multi-Stability and Chaotic Dynamics



## Generalized Heterogeneous Multi-Gap Settings: 2:1 Chains

- Consider **Heterogeneous Model** with e.g. **sphere-sphere-cylinder**

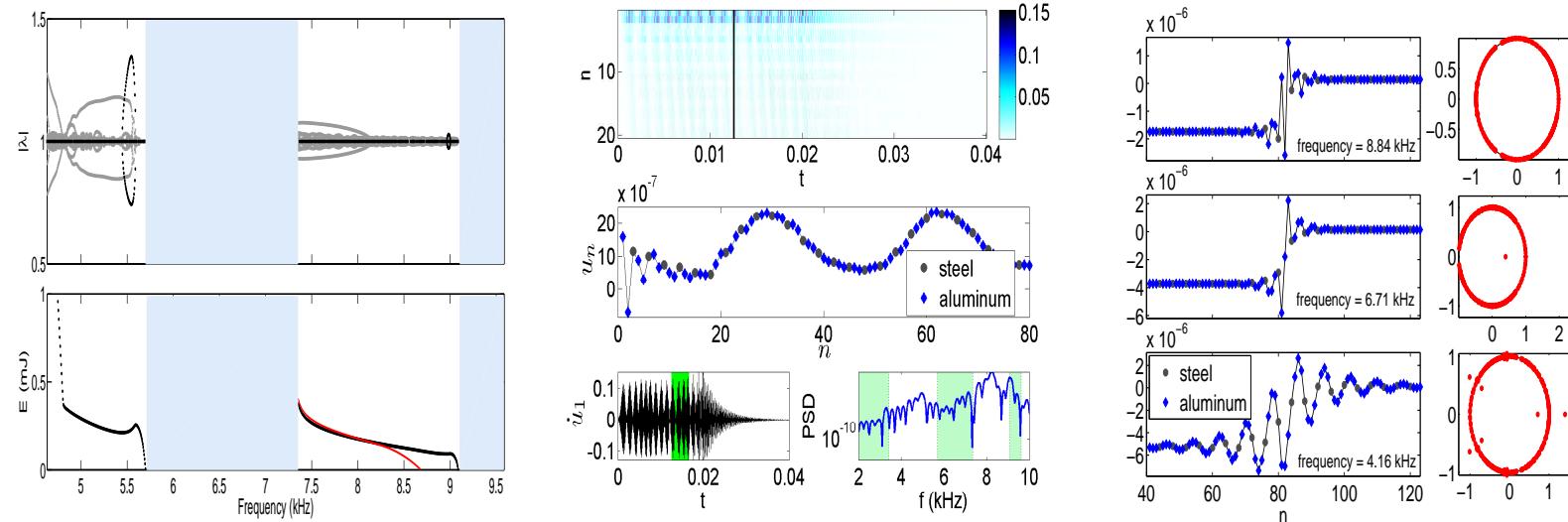
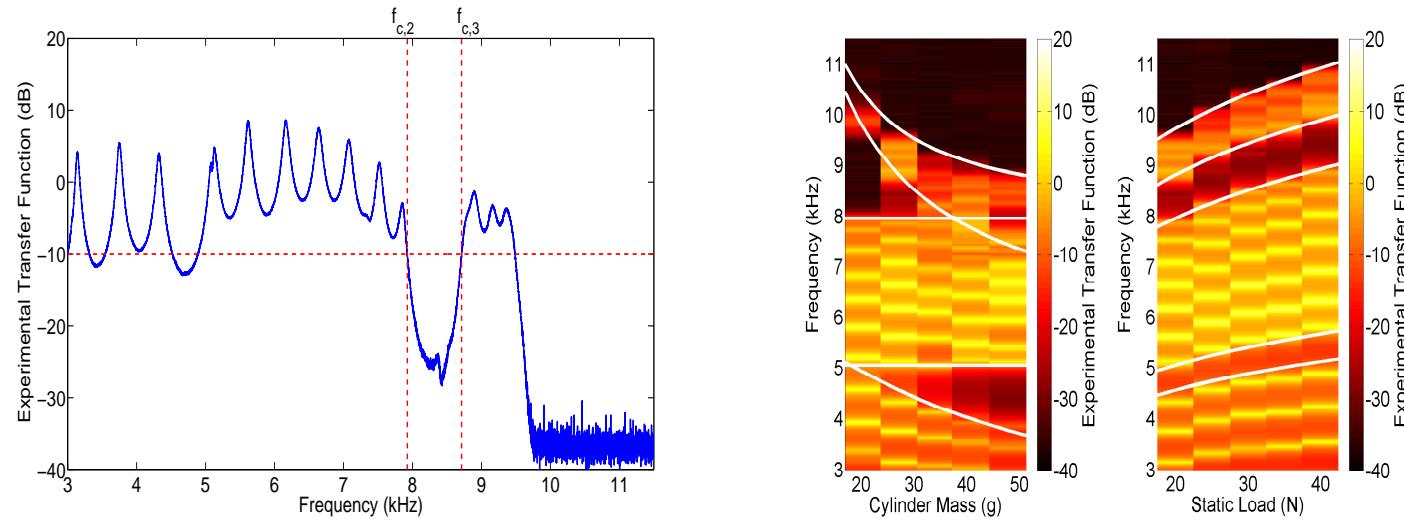
$$m_i \ddot{u}_i = A_{i-1,i} [\delta_{0,i-1,i} + u_{i-1} - u_i]_+^p - A_{i,i+1} [\delta_{0,i,i+1} + u_i - u_{i+1}]_+^p, \quad (20)$$

where  $A_1 = \frac{2E\sqrt{R}}{3(1-\nu^2)}$  and  $A_2 = \frac{E\sqrt{2R}}{3(1-\nu^2)}$ ,  $\beta_1 = \frac{3}{2}A_1^{2/3}F_0^{1/3}$  and  $\beta_2 = \frac{3}{2}A_2^{2/3}F_0^{1/3}$ .

- We can compute **Dispersion Relation** and analytically trace **Band Edges**

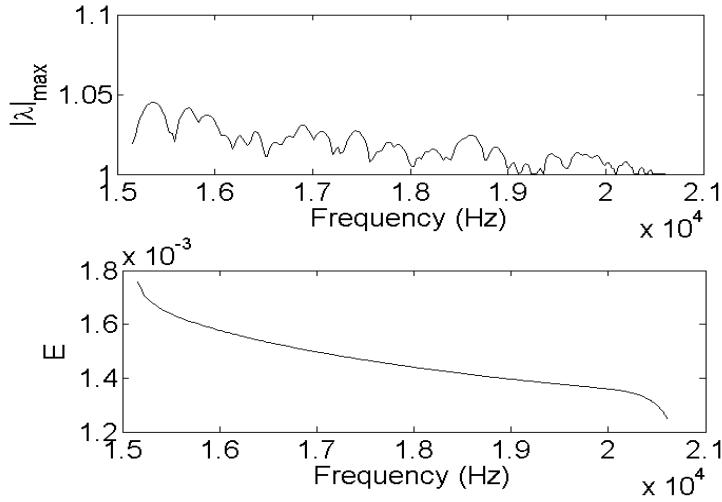
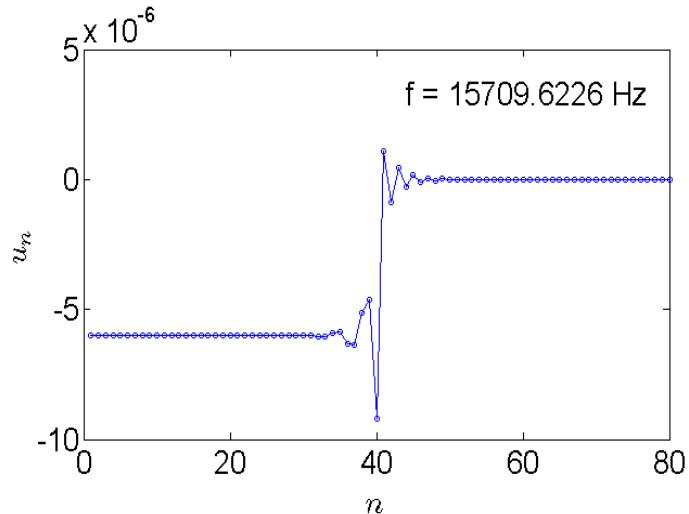
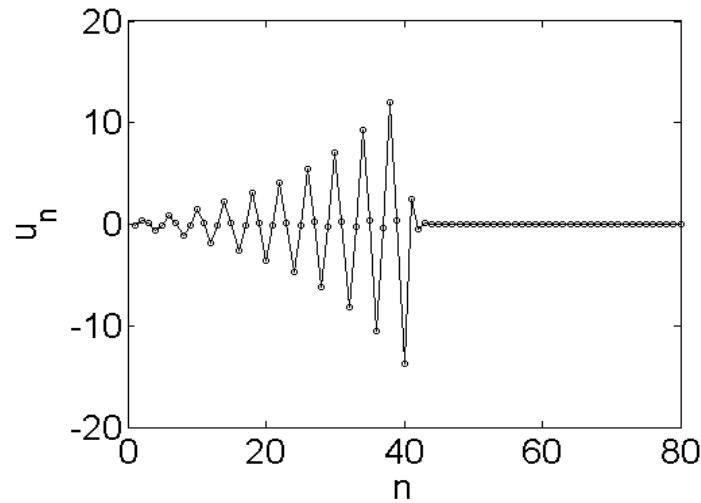
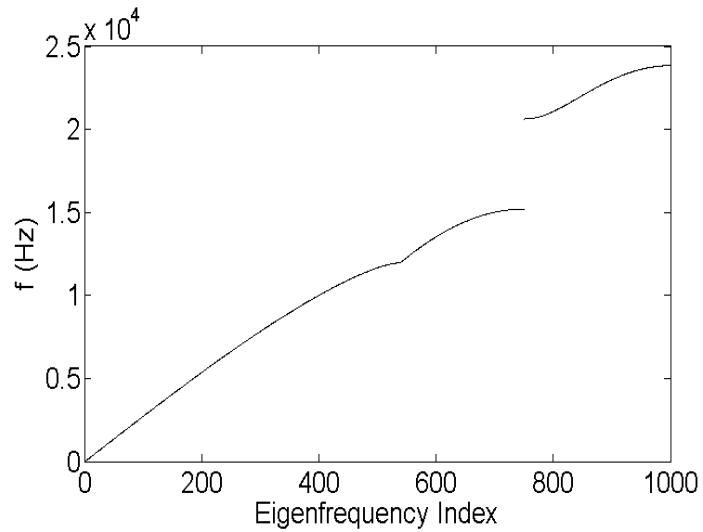
$$\begin{aligned} f_{c,1}^2 &= 0, \\ f_{c,2}^2 &= \frac{\beta_1 + 2\beta_2}{4\pi^2 m}, \\ f_{c,3}^2 &= \frac{\beta_1(2m + M)}{4\pi^2 mM}, \\ f_{c,4}^2 &= \frac{\beta_1}{4\pi^2 m}, \\ f_{c,5}^2 &= \frac{\beta_1(2m + M) + 2\beta_2 M}{8\pi^2 mM} \\ &\quad - \frac{\sqrt{-16\beta_1\beta_2 mM + (2\beta_1 m + \beta_1 M + 2\beta_2 M)^2}}{8\pi^2 mM}, \\ f_{c,6}^2 &= \frac{\beta_1(2m + M) + 2\beta_2 M}{8\pi^2 mM} \\ &\quad + \frac{\sqrt{-16\beta_1\beta_2 mM + (2\beta_1 m + \beta_1 M + 2\beta_2 M)^2}}{8\pi^2 mM}. \end{aligned} \quad (21)$$

## Experimental and Theoretical Results in Heterogeneous Chains



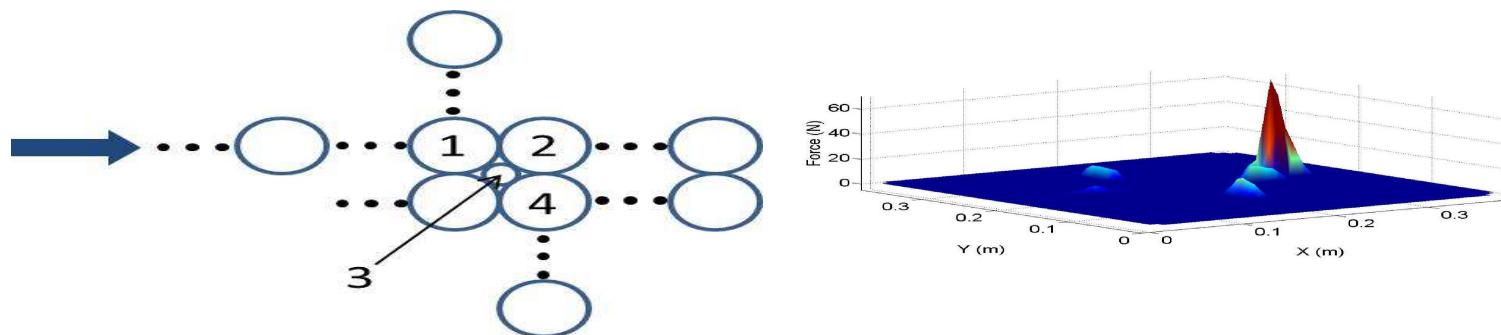
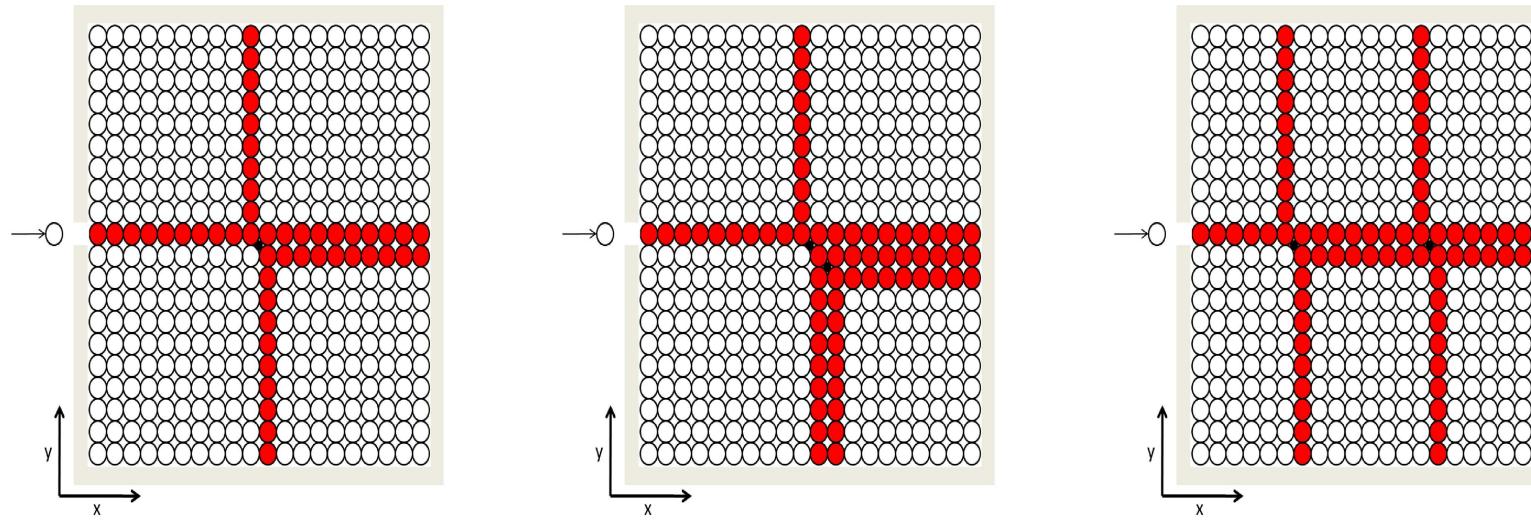
# The Role of Interfaces: Surface Nonlinear Breathing Modes

## The ABABAB...ABCCCCC Chain



## A Number of Current/Near-Future Directions

### Two-Dimensional Chains with Intruders



## Two-Dimensional Chains with Intruders (Continued)

- Find the following Energy Fractions

$$\frac{E_t}{E_i} = \frac{\frac{1}{2}mv^{(2t)^2}}{\frac{1}{2}mv^{(1i)^2}} = \left( \frac{1}{1 + (\sqrt{2} - 1)^3} \right)^2 = 87.17\% \quad (22)$$

$$\frac{E_4}{E_i} = \frac{8(mm_d)^2}{(m_d + m)^4} \sum_{k=0}^{\infty} \left( \frac{m_d - m}{m_d + m} \right)^{4k} \quad (23)$$

$$\frac{E_1}{E_i} = \frac{8(mm_d)^2}{(m_d + m)^4} \sum_{k=0}^{\infty} \left( \frac{m_d - m}{m_d + m} \right)^{4k+2}$$

	up	refl	adj	bottom	trans
Rigid particle model	3.10	2.66	3.54	3.54	87.17
Numerical	4.15	1.57	4.30	4.30	85.68

Case	up	refl	adj	adj2	bottom	bottom2	trans
Steel	4.15	1.57	4.30	0.00	4.30	0.00	85.68
Steel/steel	4.64	2.00	2.86	1.00	2.86	1.00	86.48
Steel/TC	4.69	2.05	2.53	1.28	2.53	1.28	85.65

## A Number of Current/Near-Future Directions

### Genuinely 2d Lattices: Propagation in Hexagonal Crystals

- Much more Complex Equations: Geometric Nonlinearity
- Define  $q_{m,n} = [x_{m,n}, y_{m,n}]^T$  and  $\dot{q}_{m,n} = p_{m,n} = [u_{m,n}, v_{n,m}]^T$ . Equations of Motion read:

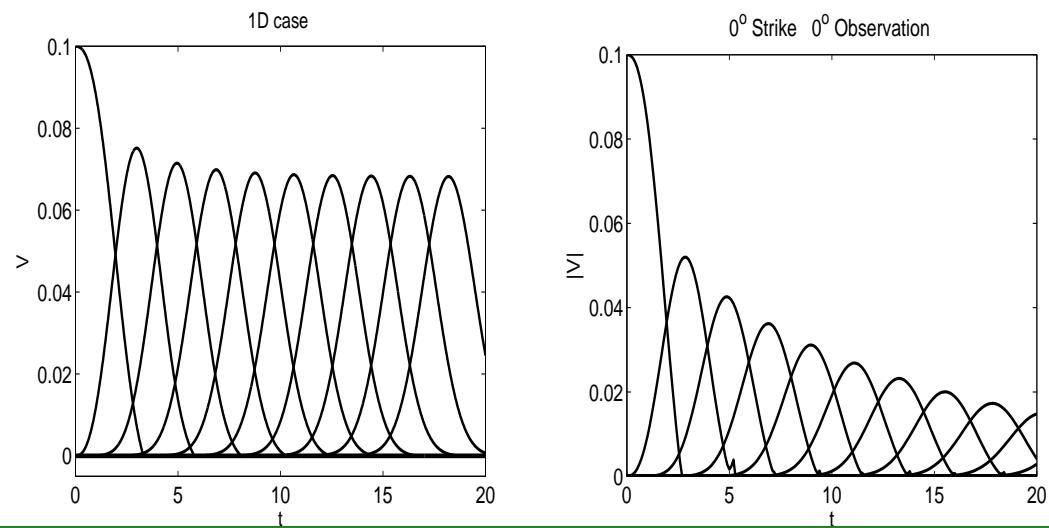
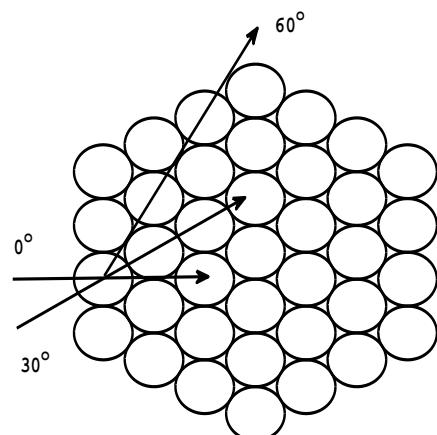
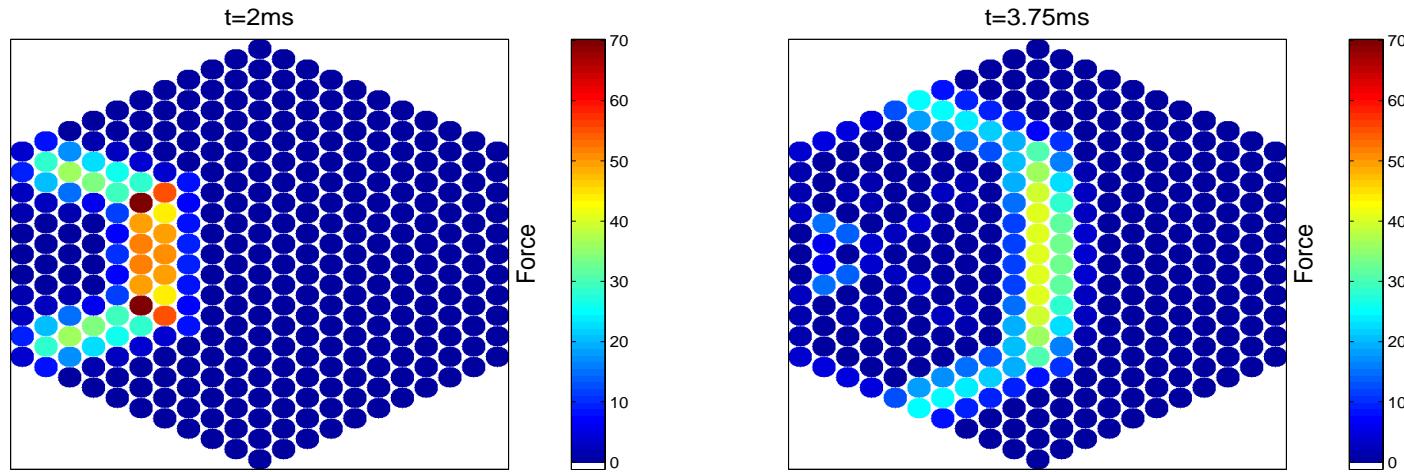
$$\left. \begin{array}{l} \dot{x}_{m,n} = u_{n,m} \\ \dot{y}_{m,n} = v_{n,m} \\ \dot{u}_{m,n} = \frac{V'(r_1) r_1^x}{r_1} + \frac{V'(r_2) r_2^x}{r_2} - \frac{V'(r_3) r_3^x}{r_3} + \frac{V'(r_4) r_4^x}{r_4} + \frac{V'(r_5) r_5^x}{r_5} - \frac{V'(r_6) r_6^x}{r_6} \\ \dot{v}_{m,n} = \frac{V'(r_1) r_1^y}{r_1} + \frac{V'(r_2) r_2^y}{r_2} - \frac{V'(r_3) r_3^y}{r_3} + \frac{V'(r_4) r_4^y}{r_4} + \frac{V'(r_5) r_5^y}{r_5} - \frac{V'(r_6) r_6^y}{r_6} \end{array} \right\} \quad (24)$$

where  $r_j = \sqrt{(r_j^x)^2 + (r_j^y)^2}$  for  $j = 1, \dots, 6$  and

$$\begin{array}{ll} r_1^x = 1 + x_{m+2,n} - x_{m,n} & r_1^y = y_{m+2,n} - y_{m,n} \\ r_2^x = \cos(\pi/3) + x_{m+1,n+1} - x_{m,n} & r_2^y = \sin(\pi/3) + y_{m+1,n+1} - y_{m,n} \\ r_3^x = \cos(\pi/3) + x_{m-1,n+1} - x_{m,n} & r_3^y = \sin(\pi/3) + y_{m-1,n+1} - y_{m,n} \\ r_4^x = \cos(\pi/3) - x_{m-2,n} + x_{m,n} & r_4^y = -\sin(\pi/3) - y_{m-2,n} + y_{m,n} \\ r_5^x = \cos(\pi/3) - x_{m-1,n-1} + x_{m,n} & r_5^y = -\sin(\pi/3) - y_{m-1,n-1} + y_{m,n} \\ r_6^x = \cos(\pi/3) - x_{m+1,n-1} + x_{m,n} & r_6^y = -\sin(\pi/3) - y_{m+1,n-1} + y_{m,n} \end{array}$$

## A Number of Current/Near-Future Directions

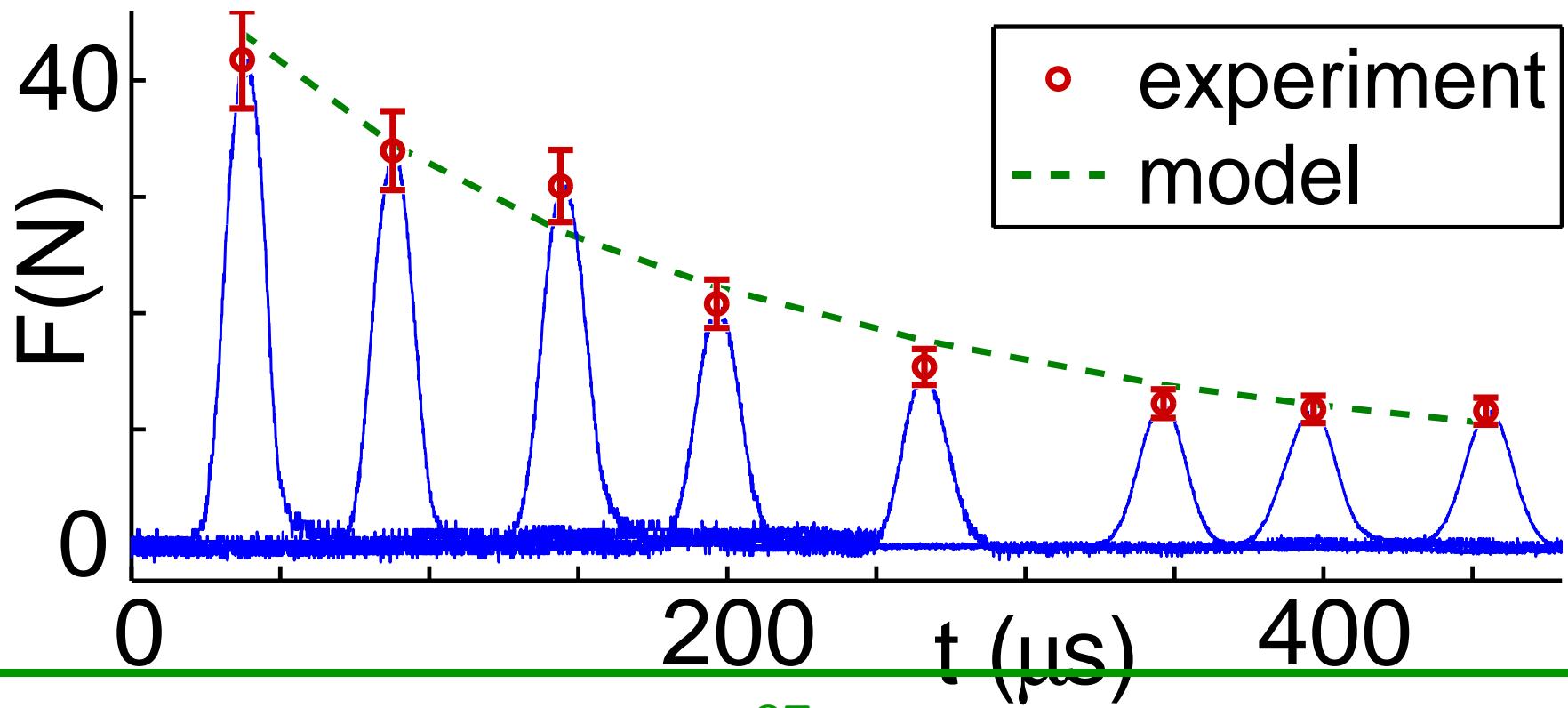
### Genuinely 2d Lattices: Propagation in Hexagonal Crystals (Continued)



## A Number of Current/Near-Future Directions

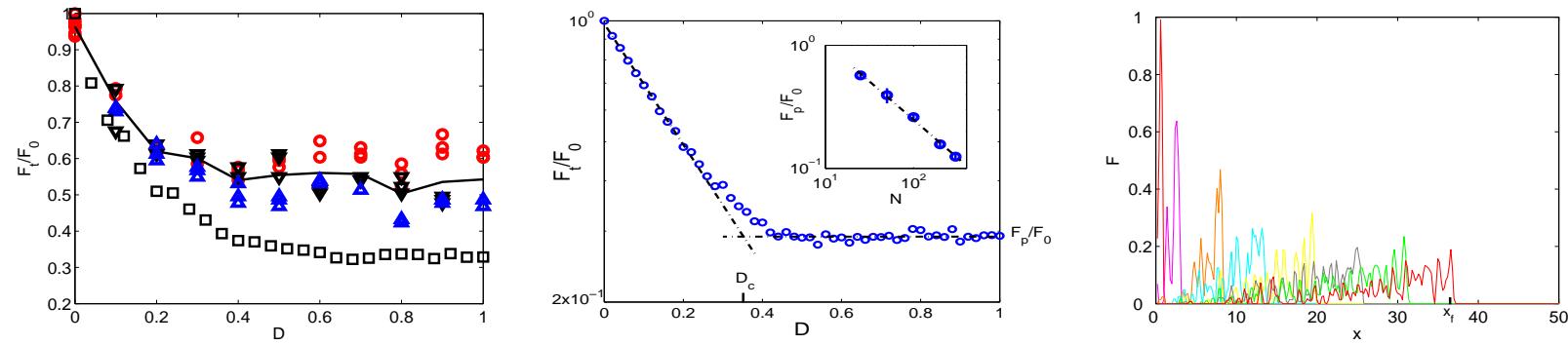
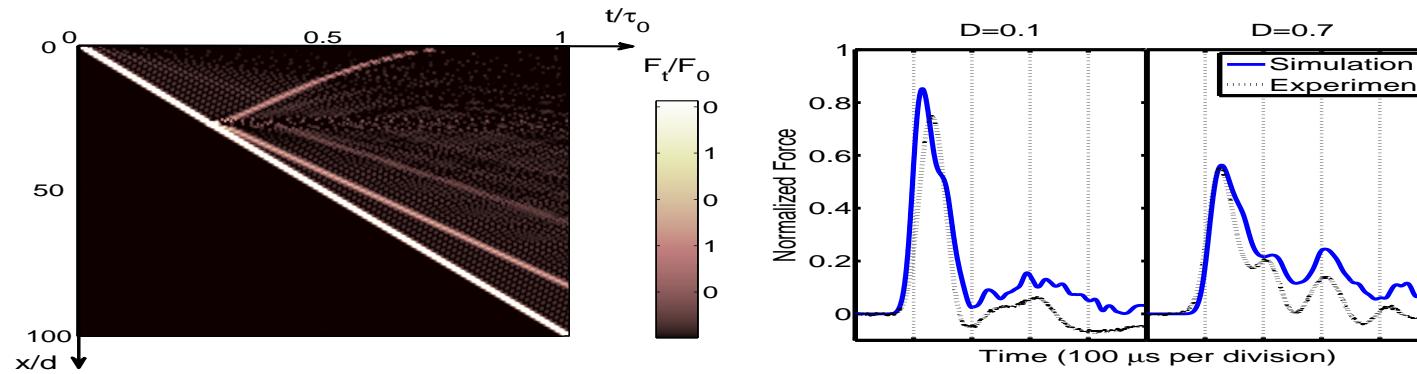
### More Elaborate Models: Inclusion of Dissipation

$$\ddot{y}_n = A \left( \delta_n^{3/2} - \delta_{n+1}^{3/2} \right) + \gamma |\dot{\delta}_n - \dot{\delta}_{n+1}|^\alpha \quad (25)$$



## The Role of Disorder: Elastic “Spin” Chains

- Consider some of the Dimer “Spins” with Reversed Polarity
- Define a (Dis)order Parameter:  $D = 1 - \frac{|N_{\text{up}} - N_{\text{down}}|}{N_{\text{up}} + N_{\text{down}}}$
- Monitor Transition from Propagation to Destruction



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## Summarizing the Picture

- Granular Crystals are a Very Rich system that can Controllably Support the excitation of a wide range of Nonlinear Beasts.
  - A Striking Excitation of the chain end typically yields Traveling Wave Solutions.
  - In the Absence of Precompression, these are Arguably the Closest Example of a Physical Realization of Nearly Compactly Supported Solution.
  - In the Presence of Precompression, these acquire Exponential Tails.
  - In the Presence of Precompression, it is also possible to induce Shock Waves, and to engineer Dark Breathers.
  - These solutions can be obtained Exactly in Numerics and observed in Experiments. They can be Analytically Approximated within the Long Wavelength Approximation.
  - In the Presence of Precompression, the system can also support Localized Breathing Modes due to Defects. These feature Interesting Bifurcations.
  - In the Dimer Limit, Modulational Instability arises, yielding Robust Discrete Breather Modes within the Linear Spectral Gap. Also, Driving & Damping yield Hysteresis Loop, Multistability & Interesting Bifurcations.
  - The Existence of such modes can be illustrated numerically (and approximately analytically via a NLS-type approximation) and they can be Unequivocally be observed in Experiments.
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## Future Outlook

- Clearly there are Numerous Directions to pursue both with respect to Traveling Waves, as well as with respect to Discrete Breathers. Some of them include:
  - A detailed look at Interface Modes and the conditions under which Stable Interfacial Energy Trapping is feasible.
  - A systematic look into Disorder Induced Transitions. Use of Scaling Theory to understand Propagation and Trapping Regimes.
  - The use of Optimization Tools/Techniques to minimize/maximize Transmitted Forces under given constraints (relevance to applications in “Granular Protectors”).
  - Stability questions arise even in the case of Traveling Waves. How does the relevant Linearization Spectrum look ?
  - Significant questions arise also for various Additional Models, such as the True Newton’s Cradle and the so-called Mass-in-Mass Model and their features.
  - All of the above questions have been posed in the One-Dimensional Setting. However, a natural challenge of this field that is presently being intensely considered concerns extensions to Higher-Dimensional Settings.
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## A Number of Current/Near-Future Directions

### More Elaborate Models: the Newton's Cradle

- Add a Local Oscillator to emulate the true Newton's Cradle
- Then, the Equation of Motion reads:

$$\ddot{y}_n + y_n = (y_{n-1} - y_n)_+^p - (y_n - y_{n+1})_+^p. \quad (26)$$

- A Multiple Scales Expansion can be made to yield:

$$2i\tau_0 \dot{A}_n = (A_{n+1} - A_n) |A_{n+1} - A_n|^{p-1} - (A_n - A_{n-1}) |A_n - A_{n-1}|^{p-1},$$

where  $\tau_0 = \frac{5(\Gamma(\frac{1}{4}))^2}{24\sqrt{\pi}} \approx 1.545$  and  $y_n^{\text{app}}(t) = 2\epsilon \operatorname{Re} [A_n(\epsilon^{1/2}t) e^{it}]$ .

- Coordinate Transformation for  $w_n = (u_{n+1} - u_n) |u_{n+1} - u_n|^{1/2}$  (and  $p = 3/2$ ) yields  $w_{n+1} - 2w_n + w_{n-1} + w_n |w_n|^{-1/3} = 0$ , and using Staggering  $w_n = (-1)^n f(n)$  and Long Wavelength Limit

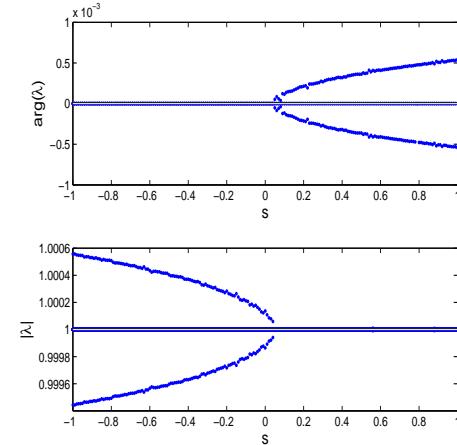
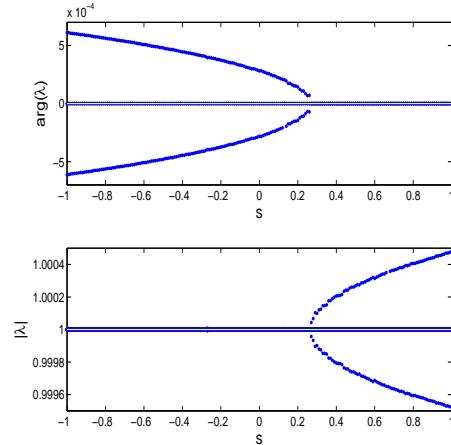
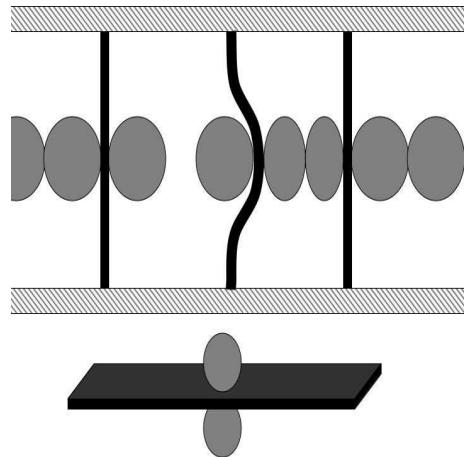
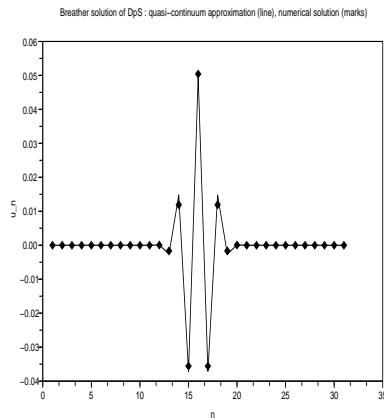
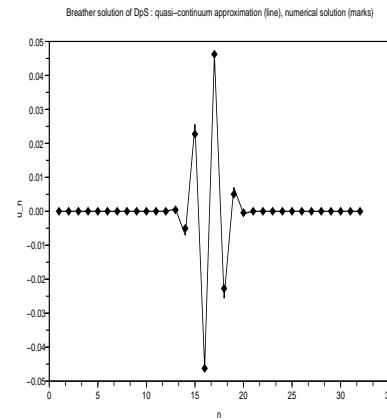
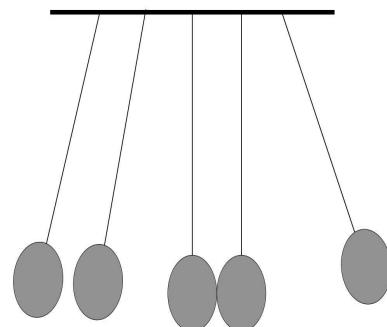
$$F'' = -4F + F |F|^{-1/3}, \quad (27)$$

which possesses a family of compactly supported solutions  $F(x) = \pm g(x + \phi)$ , where

$$g(x) = \left(\frac{3}{10}\right)^3 \cos^6\left(\frac{x}{3}\right) \text{ for } |x| \leq \frac{3\pi}{2}, \quad g = 0 \text{ elsewhere.}$$

## A Number of Current/Near-Future Directions

### More Elaborate Models: the Newton's Cradle (Continued)



## A Number of Current/Near-Future Directions

### More Elaborate Models: the Mass-in-Mass Model

- If each bead has an Internal Oscillator, then the Equations of Motion read:

$$\begin{aligned} X_{i,\tau\tau} &= \left[ (X_{i-1} - X_i)_+^{3/2} - (X_i - X_{i+1})_+^{3/2} \right] + \tilde{\kappa} (x_i - X_i) \\ \nu x_{i,\tau\tau} &= -\tilde{\kappa} (x_i - X_i) \end{aligned} \quad (28)$$

- Then for the Strain Variables, the model reads:

$$\begin{aligned} \Delta_{i,\tau\tau} &= \Delta_{(i-1),+}^{3/2} - 2\Delta_{(i),+}^{3/2} + \Delta_{(i+1),+}^{3/2} + \tilde{\kappa} (d_i - \Delta_i) \\ \nu d_{i,\tau\tau} &= -\tilde{\kappa} (d_i - \Delta_i) \end{aligned} \quad (29)$$

- One can solve for  $d_i$  and Back-Substitute in the equation for  $\Delta_i$  as:

$$\begin{aligned} \Delta_{i,tt} &= \Delta_{(i-1),+}^{3/2} - 2\Delta_{(i),+}^{3/2} + \Delta_{(i+1),+}^{3/2} + \varepsilon R(\Delta_i, t) \\ R(\Delta_i, t) &= \left( \omega_0 \int_{-\infty}^t \sin(\omega_0(t-\tau)) \Theta(t-\tau) \Delta_i(\tau) d\tau - \Delta_i \right) \end{aligned} \quad (30)$$

- Typically this leads to Decay but there is a New Possibility, namely Nanoptera. In Fourier Space, the relevant equations read:

$$\hat{R} = \frac{1 - \nu c^2 k^2}{1 + \nu - \nu c^2 k^2} \left( \frac{\sin(\frac{k}{2})}{c^{\frac{k}{2}}} \right)^2 (R^{\hat{3}/2}), \quad \hat{S} = \frac{1}{1 - \nu c^2 k^2} \hat{R}. \quad (31)$$

## A Number of Current/Near-Future Directions

### More Elaborate Models: the Mass-in-Mass Model (Continued)

