

NOTES FOR MATH 132

Sequences

Definition: A sequence is an infinite list of ordered numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

a_1 is called the first term, a_2 is the second term, in general a_n is the n th term. Since in the notion of a sequence there is a natural correspondence between the set of all positive integers (n 's) and the set of all real numbers, a sequence can be thought of as a function $f : \mathbf{N} \rightarrow \mathbf{R}$, where \mathbf{N} , the domain of f , is the set of all positive integers, and \mathbf{R} is the set of all real numbers.

Notation: $\{a_1, \dots, a_n, \dots\}$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$. Also notice that in functional notation $a_n = f(n)$.

Limit of a Sequence

Definition: If an infinite sequence of numbers $a_1, a_2, \dots, a_n, \dots$ is given, and there is a number L such that every interval, no matter how small, marked off about the point L contains all the points a_n except for at most a finite number, we say that the number L is the limit of the sequence a_1, \dots, a_n, \dots , that is the sequence a_1, \dots, a_n, \dots converges to L . In symbols $\lim_{n \rightarrow \infty} a_n = L$.

An equivalent definition can be given using the so called $\epsilon - \delta$ language.

Definition: A sequence $\{a_n\}$ has the limit L and we write $\lim_{n \rightarrow \infty} a_n = L$ or $a_n \rightarrow L$ as $n \rightarrow \infty$, if for every $\epsilon > 0$ there is a corresponding integer N such that $|a_n - L| < \epsilon$ whenever $n > N$. If $\lim_{n \rightarrow \infty} a_n$ exists, we say that the sequence $\{a_n\}$ converges (or is convergent), otherwise we say that the sequence diverges (or is divergent).

Note: Dropping (the first) finite number of terms of a sequence does not affect the convergence.

Using the functional notation for a sequence it is easy to see the following fact.

Theorem: If $\lim_{x \rightarrow \infty} f(x) = L$, and $a_n = f(n)$ when n is an integer, then $\lim_{n \rightarrow \infty} a_n = L$.

This Theorem gives you a method of convergence analysis for sequences using the known results for limits of functions (*e.g.* L'Hopital's rule).

If a_n becomes large as n becomes large, we use the notation $\lim_{n \rightarrow \infty} a_n = \infty$, the meaning of which is given by

Definition: $\lim_{n \rightarrow \infty} a_n = \infty$ means that for every number M there is an integer N such that

$$a_n > M \quad \text{whenever} \quad n > N.$$

Note: In the case $\lim_{n \rightarrow \infty} a_n = \infty$ the sequence is still divergent and it is said to diverge to ∞ .

Properties of Limits

$\{a_n\}$ and $\{b_n\}$ are convergent sequences, c is a constant

$$1. \lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n.$$

$$2. \lim_{n \rightarrow \infty} (ca_n) = c \cdot \lim_{n \rightarrow \infty} a_n.$$

$$3. \lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n.$$

$$4. \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \quad \text{if} \quad \left(\lim_{n \rightarrow \infty} b_n \neq 0 \right).$$

$$5. \lim_{n \rightarrow \infty} c = c.$$

6. The squeeze Theorem

If $a_n \leq b_n \leq c_n$ for $n > n_0$, and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.

Monotonic Sequences

Definition: A sequence $\{a_n\}$ is called increasing if $a_n \leq a_{n+1}$ for all $n \geq 1$, that is $a_1 \leq a_2 \leq \dots \leq a_n \leq \dots$.

It is called decreasing if $a_n \geq a_{n+1}$ for all $n \geq 1$. It is called monotonic, if it is either increasing or decreasing.

Definition: A sequence $\{a_n\}$ is bounded above, if there is a number M such that $a_n \leq M$ for all $n \geq 1$.

It is bounded below, if there is a number m such that $m \leq a_n$ for all $n \geq 1$.

If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.

Notice that a more general result holds, if we only require the sequence to be monotonic beginning from some index n_0 .

Series

Definition: An expression of the form

$$a_1 + a_2 + \cdots + a_n + \cdots$$

is called an infinite series (or just a series), and for short, is denoted $\sum_{n=1}^{\infty} a_n$.

As the definition suggests the series is a generalization of a finite sum to the case of infinitely many summands. To investigate whether it makes sense to talk about a sum of infinitely many terms we will start by considering partial sums.

n th partial sum, denoted by s_n , is the sum of the first n terms of the series

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^n a_i$$

Notice that $s_1, s_2, \dots, s_n, \dots$ form a sequence of numbers called sequence of partial sums for a series.

Definition: Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$, if the sequence of the partial sums $\{s_n\}$ is convergent, and $\lim_{n \rightarrow \infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the sum of the series.

Otherwise (if $\{s_n\}$ is divergent) the series is called divergent.

Theorem (Necessary condition for convergence): If the series $\sum a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$.

Using this fact it's easy to formulate the following test

The Test for Divergence: If $\lim_{n \rightarrow \infty} a_n$ does not exist or $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum a_n$ is divergent.

Properties of Convergent Series

If $\sum a_n$ and $\sum b_n$ are convergent series then so are the series $\sum ca_n$ (where c is a constant), and $\sum(a_n \pm b_n)$, and

$$1. \sum_{n=1}^{\infty} ca_n = c \cdot \sum_{n=1}^{\infty} a_n$$

$$2. \sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n.$$

Note: Even if $\sum (a_n b_n)$ is convergent (which is not true in general)

$$\sum (a_n b_n) \neq \left(\sum a_n \right) \cdot \left(\sum b_n \right).$$

Convergence Tests for Positive Series

The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1; \infty)$, and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if

the improper integral $\int_1^{\infty} f(x) dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x) dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x) dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Notice that since dropping first finite number of terms doesn't affect the convergence of a series, in the Integral Test it is not necessary to start the series or integral at $n = 1$.

Definition: $R_n = s - s_n = a_{n+1} + a_{n+2} + \dots$ is called the n th remainder of the series $\sum a_n$. For short

$$R_n = \sum_{i=n+1}^{\infty} a_i$$

The remainder basically measures how far the partial sums are from the sum of the series. Using the notion of the remainder we have the following error estimate in the Integral Test.

Remainder Estimate for the Integral Test: If $\sum a_n$ converges by the Integral Test, and $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$$

or replacing R_n by $s - s_n$ one easily has the following estimate for the sum of the series

$$s_n + \int_{n+1}^{\infty} f(x) dx \leq s \leq s_n + \int_n^{\infty} f(x) dx.$$

The Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- (i) if $\sum b_n$ is convergent and $a_n \leq b_n$ for all n then $\sum a_n$ is also convergent.
- (ii) if $\sum b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum a_n$ is also divergent.

A slight modification of the Comparison Test using the definition of the limit is given by the following

The Limit Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim \frac{a_n}{b_n} = c$$

where c is finite, and $c > 0$, then either both series converge, or both series diverge, which is

- (i) if $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.
- (ii) if $\sum b_n$ is divergent, then $\sum a_n$ is also divergent.

Alternating Series

Definition: An alternating series is a series whose terms are alternating positive or negative, *i.e.* $\sum a_n$ is an alternating series if $a_n = (-1)^{n-1}b_n$ for $n = 1, 2, \dots$, where b_n is a positive number (in fact $b_n = |a_n|$).

Alternating Series Test: If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - \dots b_n \dots \quad \text{where } b_n \geq 0$$

satisfies

- (i) $b_{n+1} \leq b_n$ for all n ($\{b_n\}$ is increasing)
- (ii) $\lim_{n \rightarrow \infty} b_n = 0$

then the series is convergent.

Using the associativity of addition, a skillful placement of parenthesis gives the following estimation result.

Alternating Series Estimation Theorem: If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

- (i) $0 \leq b_{n+1} \leq b_n$ for all n ($\{b_n\}$ is increasing)
- (ii) $\lim_{n \rightarrow \infty} b_n = 0$

then $|R_n| = |s - s_n| \leq b_{n+1}$.

Absolute and Conditional Convergence

Definition: A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

A series which is convergent, but not absolutely convergent, is called conditionally convergent.

Notice that absolutely convergent series are convergent.

Theorem: If $\sum a_n$ is absolutely convergent, then it is convergent.

The absolute or conditional convergence properties of series indicate whether infinite sums behave like finite sums or no. It turns out that a rearrangement of terms of an absolutely convergent series has no impact on its sum,

If $\sum a_n = s$ where $\sum a_n$ is absolutely convergent, then any rearrangement of $\sum a_n$ has the same sum s .

However by Rieman's result a skillful rearrangement of a conditionally convergent series can change its sum to any beforehand given number,

If $\sum a_n$ is conditionally convergent series, and r is any real number ($r \in \mathbf{R}$), then there is a rearrangement of $\sum a_n$ that has a sum equal to r .

Absolute Convergence Tests

The Ratio Test:

(i) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: In the cases where $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ does not exist, or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test doesn't give any answer about the behaviour of the series, and other convergence tests must be considered.

The Root Test:

(i) If $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = L > 1$ or $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: In the cases where $\lim_{n \rightarrow \infty} \sqrt[n]{a_n}$ does not exist, or $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = 1$, the Root Test doesn't give any answer about the behaviour of the series, and other convergence tests must be considered.

Notice also that as the proofs suggest, the Ratio and Root Tests are just convenient reformulations of the Comparison Test, where series are compared with the geometric series.

Power Series

Definition: A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

where x is a variable, and the c 's are constants called the coefficients of the series.

For each fixed value of x , the power series becomes just a number series for which the question of convergence can be studied.

More generally a series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots + c_n (x - a)^n + \cdots$$

where a is a number, is called a power series in $(x - a)$, or a power series about a , or a power series centered at a .

Notice that for convenience of notation $(x - a)^0$ is taken to be 1 even when $x = a$.

Investigating the possible values of x for which the power series converges, it turns out that there aren't very many possibilities, which is given by the following.

Theorem: For a given power series $\sum_{n=0}^{\infty} c_n (x - a)^n$ there are only three possibilities:

- (i) The series converges only when $x = a$
- (ii) The series converges for all x ($\forall x \in \mathbf{R}$)
- (iii) There is a positive number R such that the series converges if $|x - a| < R$ and diverges for $|x - a| > R$.

The number R in case (iii) is called radius of convergence of the power series. By convention $R = 0$ in case (i), and $R = \infty$ in case (ii).

Differentiation and Integration of Power series

For all those values of x for which $\sum c_n (x - a)^n$ converges, the sum of the series can be defined, and will be a function of x

Theorem: If the power series $\sum c_n (x - a)^n$ has radius of convergence $R > 0$, then the function defined by

$$f(x) = c_0 + c_1 (x - a) + \cdots + c_n (x - a)^n + \cdots = \sum_{n=0}^{\infty} c_n (x - a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R; a + R)$ and

$$(i) \quad f'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$(ii) \quad \int f(x)dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + \cdots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.$$

The radii of convergence of the power series in equations (i) and (ii) are both R .

Taylor and Maclaurin Series

Theorem: If f has a power series representation (expansion) at a , that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting this formula in the expansion of f , we see that if f can be expanded in a power series about a , then the power series necessarily is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + \frac{f'(a)}{1!} (x-a) + \cdots + \frac{f^{(n)}(a)}{n!} (x-a)^n + \cdots.$$

This series is called the Taylor series of the function f at a . For the special case $a = 0$ Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots$$

which is called the Maclaurin series of the function f .

Using the formulas given by the above theorem one can construct the Taylor series for every infinitely differentiable function, but in general f may not be equal to the sum of its Taylor series.

Theorem: If $f(x) = T_n(x) + R_n(x)$, where T_n is the n th partial sum of the Taylor series about a (also called the n th Taylor polynomial of f), and

$$\lim_{n \rightarrow \infty} R_n(x) = 0 \quad \text{for } |x-a| < R$$

then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

In trying to show that $\lim_{n \rightarrow \infty} R_n(x) = 0$ for a specific function f , the following result can be used.

Taylor's Inequality: If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$ then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$R_n(x) \leq \frac{M}{(n+1)!} |x - a|^{n+1} \quad \text{for } |x - a| \leq d.$$

APPENDIX

The sequence r^n is convergent if $|r| < 1$ and is divergent for all other values of r

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \end{cases}.$$

The Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^n + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad |r| < 1$$

If $|r| \geq 1$ the series is divergent.

The p -series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Maclaurin Series of some elementary functions

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

$$\ln(1-x) = - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$