# An Augmented Harmonic Lanczos Bidiagonalization Method Applied to Least Squares Problems 

James Baglama, Lothar Reichel, Dan Richmond

University of Rhode Island
NENAD 2012
UMass Amherst
April 21, 2012

## Linear Algebra Background

■ Sparse matrix: Matrix where most entries are zero

- Orthogonal: $V^{T} V=1$
- Column rank: Number of linearly independent column vectors of a matrix
- QR decomposition: $A=Q R$ where $Q^{T} Q=I, R$ upper triangular

■ Krylov subspace: $\mathbb{K}_{j}(A, v)=\operatorname{span}\left\{v, A v, A^{2} v, \ldots, A^{j-1} v\right\}$

## Linear Algebra Background

- Singular Value Decomposition (SVD): Let $A \in \mathbb{R}^{\ell \times n}$ with $\ell \geq n$.

Then we can write $A=U \Sigma V^{T}$ with $U \in \mathbb{R}^{\ell \times n}$,

$$
\Sigma=\operatorname{diag}\left[\sigma_{1}, \sigma_{2}, \ldots, \sigma_{n}\right] \in \mathbb{R}^{n \times n}, V \in \mathbb{R}^{n \times n} \text { such that }
$$

$$
U^{T} U=I_{n}, V^{T} V=I_{n}
$$

- $U$ is the matrix of left singular vectors
- $V$ is the matrix of right singular vectors
- $\Sigma$ is the diagonal matrix of singular values


## Lanczos Bidiagonalization

$m$ step Lanczos Bidiagonalization

$$
\begin{aligned}
A^{T} Q_{m+1} & =P_{m} B_{m+1, m}^{T}+\alpha_{m+1} p_{m+1} e_{m+1}^{T} \\
A P_{m} & =Q_{m+1} B_{m+1, m}
\end{aligned}
$$

$P_{m}=\left[p_{1}, \ldots, p_{m}\right] \in \mathbb{R}^{n \times m}$ : Orthogonal matrix
$Q_{m+1}=\left[q_{1}, \ldots, q_{m+1}\right] \in \mathbb{R}^{\ell \times m+1}$ : Orthogonal matrix
$B_{m+1, m} \in \mathbb{R}^{m+1 \times m}$ : Lower bidiagonal matrix
$p_{m+1} \in \mathbb{R}^{n}:$ Residual vector
$\alpha_{m+1} \in \mathbb{R}$
(Golub, Kahan 1965)


1. Compute $\beta_{1}=\left\|q_{1}\right\| ; q_{1}=q_{1} / \beta_{1}$
2. Compute $p_{1}=A^{T} q_{1} ; \alpha_{1}=\left\|p_{1}\right\| ; p_{1}=p_{1} / \alpha_{1}$ for $j=1: m$
3. Compute $q_{j+1}=A p_{j}-q_{j} \alpha_{j}$

3a. Reorthogonalization: $q_{j+1}=q_{j+1}-Q_{(1: j)}\left(Q_{(1: j)}^{T} q_{j+1}\right)$
4. Compute $\beta_{j+1}=\left\|q_{j+1}\right\| ; q_{j+1}=q_{j+1} / \beta_{j+1}$
5. Compute $p_{j+1}=A^{T} q_{j+1}-p_{j} \beta_{j+1}$

5a. Reorthogonalization: $p_{j+1}=p_{j+1}-P_{(1: j)}\left(P_{(1: j)}^{T} p_{j+1}\right)$
6. Compute $\alpha_{j+1}=\left\|p_{j+1}\right\| ; p_{j+1}=p_{j+1} / \alpha_{j+1}$

■ Solve the least squares problem

$$
\min _{x \in \mathbb{R}^{n}}\|b-A x\|_{2}
$$

- $A \in \mathbb{R}^{\ell \times n}$ is a large sparse matrix
- Requires an iterative method
- Assume $A$ has full column rank

■ Iterative algorithms: GMRES, CG, CGNR, LSQR, LSMR
■ Improve the convergence speed of LSQR

## Outline of LSQR

- Based on the Lanczos Bidiagonalization method
- Iteration begins:
- Initial guess $x_{0}$ and initial residual $r_{0}=b-A x_{0}$
- Set $q_{1}=r_{0} /\left\|r_{0}\right\|_{2}$ and $p_{1}=A^{T} q_{1} /\left\|A^{T} q_{1}\right\|_{2}$
- Process continues:
- Generates an orthonormal set of vectors $\left\{q_{1}, q_{2}, \ldots, q_{m}\right\}$ for

$$
\mathbb{K}_{m}\left(A A^{T}, q_{1}\right)=\operatorname{span}\left\{q_{1}, A A^{T} q_{1},\left(A A^{T}\right)^{2} q_{1}, \ldots,\left(A A^{T}\right)^{m-1} q_{1}\right\}
$$

- Generates an orthonormal set of vectors $\left\{p_{1}, p_{2}, \ldots, p_{m}\right\}$ for

$$
\mathbb{K}_{m}\left(A^{T} A, p_{1}\right)=\operatorname{span}\left\{p_{1}, A^{T} A p_{1},\left(A^{T} A\right)^{2} p_{1}, \ldots,\left(A^{T} A\right)^{m-1} p_{1}\right\}
$$

## Outline of LSQR

- Approximate solution $x_{m} \in x_{0}+\mathbb{K}_{m}\left(A^{T} A, p_{1}\right)$
- Residual vector $r_{m}=b-A x_{m} \in \mathbb{K}_{m}\left(A A^{T}, q_{1}\right)$


## Outline of LSQR

- Approximate solution $x_{m} \in x_{0}+\mathbb{K}_{m}\left(A^{T} A, p_{1}\right)$
- Residual vector $r_{m}=b-A x_{m} \in \mathbb{K}_{m}\left(A A^{T}, q_{1}\right)$
- Let $A=U_{n} \Sigma_{n} V_{n}^{T}$ be the SVD of $A$
- $U_{n} \in \mathbb{R}^{\ell \times n}$ and $V_{n} \in \mathbb{R}^{n \times n}$ orthogonal matrices
- $\Sigma_{n}=\operatorname{diag}\left[\sigma_{1}, \ldots, \sigma_{n}\right] \in \mathbb{R}^{n \times n}$ where $0<\sigma_{1} \leq \sigma_{2} \leq \ldots \leq \sigma_{n}$
- The $m^{\text {th }}$ residual of LSQR satisfies (Bjork, 1990):

$$
\left\|r_{m}-r^{+}\right\|_{2} \leq 2\left(\frac{\sigma_{n}-\sigma_{1}}{\sigma_{n}+\sigma_{1}}\right)^{m}\left\|r_{0}-r^{+}\right\|_{2}
$$

where $r^{+}=b-A A^{+} b$

## Convergence Properties

- LSQR can exhibit slow convergence for ill-conditioned matrices and when the solution vector has components in the direction of the singular vectors associated with the smallest singular values (Bjork, 1990).


## Convergence Properties

■ LSQR can exhibit slow convergence for ill-conditioned matrices and when the solution vector has components in the direction of the singular vectors associated with the smallest singular values (Bjork, 1990).

- We have developed a preconditioned LSQR method that during preconditioning steps computes the solution and reduces the upper bound on the residual norm simultaneously. After an acceptable number of preconditioning steps, the method reverts to LSQR until convergence.


## Using an Augmented Krylov Subspace

The idea is to compute an approximate solution $x_{m}$ and residual $r_{m}$ from augmented Krylov subspaces respectively:
$\mathbb{K}_{m}\left(A A^{T}, u_{1}, \ldots, u_{k}, q_{1}\right)=\operatorname{span}\left\{u_{1}, \ldots, u_{k}, q_{1}, A A^{T} q_{1}, \ldots,\left(A A^{T}\right)^{m-k-1} q_{1}\right\}$
$\mathbb{K}_{m}\left(A^{T} A, v_{1}, \ldots, v_{k}, p_{1}\right)=\operatorname{span}\left\{v_{1}, \ldots, v_{k}, p_{1}, A^{T} A p_{1}, \ldots,\left(A^{T} A\right)^{m-k-1} p_{1}\right\}$
where $u_{1}, \ldots, u_{k}$ and $v_{1}, \ldots, v_{k}$ are the left and right singular vectors corresponding to the $k$ smallest singular values of $A$, respectively.

- Theorem 1. Let $A \in \mathbb{R}^{\ell \times n}$ with the singular value decompositions as before and extract the minimum residual solution $x_{m}$ from the space $x_{0}+\mathbb{K}_{m}\left(A^{T} A, v_{1}, \ldots, v_{k}, p_{1}\right)$ then

$$
\left\|r_{m}-r^{+}\right\|_{2} \leq 2\left(\frac{\sigma_{n}-\sigma_{k+1}}{\sigma_{n}+\sigma_{k+1}}\right)^{m-k}\left\|r_{0}-r^{+}\right\|_{2}
$$

Proof: (Baglama, Reichel, Richmond, 2012)

## Augmenting with Singular Vectors




Example of Theorem 1. $A$ is the $1850 \times 712$ matrix ILLC1850 and the right-hand side $b$ is ILLC1850_RHS1 from the Matrix Market Collection.

## Motivation

- Singular values/vectors not known prior to the start.
- Singular values/vectors not known prior to the start.

■ How to generate good approximations to the singular values/vectors while simultaneously updating the solution?

- Singular values/vectors not known prior to the start.
- How to generate good approximations to the singular values/vectors while simultaneously updating the solution?
- We use a restarted LSQR method augmented with Harmonic Ritz vectors. We can restart on the augmented space since the residual of LSQR and the residual of the Harmonic Ritz vectors are multiples of the same vector.
- Singular values/vectors not known prior to the start.
- How to generate good approximations to the singular values/vectors while simultaneously updating the solution?
- We use a restarted LSQR method augmented with Harmonic Ritz vectors. We can restart on the augmented space since the residual of LSQR and the residual of the Harmonic Ritz vectors are multiples of the same vector.

■ Morgan (1991, 2000, 2002) implemented a similar idea using a restarted GMRES method augmented with approximate eigenvectors for solving linear systems.

- Theorem 2. Let $A \in \mathbb{R}^{\ell \times n}$ with the singular value decompositions as before and extract the minimum residual solution $x_{m}$ of from the augmented Krylov subspace $x_{0}+\mathbb{K}_{m}\left(A^{T} A, y_{1}, p_{1}\right)$ where $y_{1}$ is an approximate right singular vector to $v_{1}$. Let $\zeta$ represent the angle between $y_{1}$ and $v_{1}$, then

$$
\left\|r_{m}-r^{+}\right\| \leq 2\left(\frac{\sigma_{n}-\sigma_{2}}{\sigma_{n}+\sigma_{2}}\right)^{m-1}\left\|r_{0}-r^{+}\right\|+\frac{\left\|A^{T} A\right\|_{2}}{\sigma_{1}} \tan (\zeta) \cdot\left|\omega_{1}\right|
$$

$\omega_{1}$ is the coefficient of $u_{1}$ in the expansion of $b$ in terms of the left singular vectors Proof: (Baglama, Reichel, Richmond, 2012)

## Lanczos Bidiagonalization

$m$ step Lanczos Bidiagonalization

$$
\begin{aligned}
A^{T} Q_{m+1} & =P_{m} B_{m+1, m}^{T}+\alpha_{m+1} p_{m+1} e_{m+1}^{T} \\
A P_{m} & =Q_{m+1} B_{m+1, m}
\end{aligned}
$$

$P_{m}=\left[p_{1}, \ldots, p_{m}\right] \in \mathbb{R}^{n \times m}$ : Orthogonal matrix
$Q_{m+1}=\left[q_{1}, \ldots, q_{m+1}\right] \in \mathbb{R}^{\ell \times m+1}$ : Orthogonal matrix
$B_{m+1, m} \in \mathbb{R}^{m+1 \times m}$ : Lower bidiagonal matrix
$p_{m+1} \in \mathbb{R}^{n}:$ Residual vector
$\alpha_{m+1} \in \mathbb{R}$
(Golub, Kahan 1965)

## Connection to LBD for $A A^{T}$

- The connection to the Lanczos decomposition for $A A^{T}$
$A\left(A^{T} Q_{m+1}=P_{m} B_{m+1, m}^{T}+\alpha_{m+1} p_{m+1} e_{m+1}^{T}\right)$
$A A^{T} Q_{m+1}=Q_{m+1} B_{m+1, m} B_{m+1, m}^{T}+\alpha_{m+1} A p_{m+1} e_{m+1}^{T}$
$A p_{m+1}$ is unknown, equating just the first $m$ columns gives:

$$
\begin{equation*}
A A^{T} Q_{m}=Q_{m} B_{m} B_{m}^{T}+\alpha_{m} \beta_{m+1} e_{m}^{T} \tag{1}
\end{equation*}
$$

- The Harmonic Ritz values, $\theta_{j}$, of (1) are the eigenvalues of the eigenvalue problem (Paige et. al 1993):

$$
\left(\left(B_{m} B_{m}^{T}\right)+\alpha_{m}^{2} \beta_{m+1}^{2}\left(B_{m} B_{m}^{T}\right)^{-1} e_{m} e_{m}^{T}\right) g_{j}=\theta_{j} g_{j}
$$

- Compute the Harmonic Ritz values without forming $B_{m} B_{m}^{T}$

■ Let $B_{m+1, m}=\tilde{U}_{m+1, m} \tilde{\Sigma}_{m} \tilde{V}_{m}^{T}$ be the SVD of $B_{m+1, m}$

- $\tilde{V}_{m} \in \mathbb{R}^{m \times m}$ and $\tilde{U}_{m+1, m} \in \mathbb{R}^{m+1 \times m}$ orthogonal matrices
- $\tilde{\Sigma}_{m}=\operatorname{diag}\left[\tilde{\sigma}_{1}, \ldots, \tilde{\sigma}_{m}\right] \in \mathbb{R}^{m \times m}$ where $0<\tilde{\sigma}_{1} \leq \tilde{\sigma}_{2} \leq \ldots \leq \tilde{\sigma}_{m}$

■ Harmonic Ritz values of $A A^{T} Q_{m}=Q_{m} B_{m} B_{m}^{T}+\alpha_{m} \beta_{m+1} e_{m}^{T}$

$$
\theta_{j}=\tilde{\sigma}_{j}^{2}
$$

## Harmonic Ritz Vectors

- Eigenvectors of

$$
\begin{gathered}
\left(\left(B_{m} B_{m}^{T}\right)+\alpha_{m}^{2} \beta_{m+1}^{2}\left(B_{m} B_{m}^{T}\right)^{-1} e_{m} e_{m}^{T}\right) g_{j}=\theta_{j} g_{j} \\
{\left[g_{1}, \ldots, g_{m}\right]=\left(I_{m} \quad \beta_{m+1} B_{m}^{-T} e_{m}\right) \tilde{U}_{m+1, m}}
\end{gathered}
$$

- The Harmonic Ritz vector of $A A^{T}$ associated with the Harmonic Ritz value of $\theta_{j}=\tilde{\sigma}_{j}^{2}$ is defined as:

$$
\hat{u}_{j}=Q_{m} g_{j}
$$

## Harmonic Ritz Vectors

■ Residual errors associated with different Harmonic Ritz pairs $\left\{\theta_{j}, \hat{u}_{j}\right\}$ are multiples of the same vector

- $A A^{T} \hat{u}_{j}-\theta_{j} \hat{u}_{j}=\left(\alpha_{m} \beta_{m+1} e_{m}^{T} g_{j}\right) Q_{m+1}\binom{-\beta_{m+1} B_{m}^{-T} e_{m}}{1}$
- Define $r_{m}^{\text {harm }}=Q_{m+1}\binom{-\beta_{m+1} B_{m}^{-T} e_{m}}{1}$


## Stability

- Accurate computation of $B_{m}^{-T} e_{m}$ can be difficult when $B_{m}^{T}$ has a large condition number

■ $B_{m+1, m}^{T}=B_{m}^{T}\left[I_{m} \beta_{m+1} B_{m}^{T} e_{m}\right]$

## Stability

- Accurate computation of $B_{m}^{-T} e_{m}$ can be difficult when $B_{m}^{T}$ has a large condition number
- $B_{m+1, m}^{T}=B_{m}^{T}\left[I_{m} \quad \beta_{m+1} B_{m}^{T} e_{m}\right]$
- $\left[\begin{array}{c}-\beta_{m+1} B_{m}^{-T} e_{m} \\ 1\end{array}\right]$ is in the null space of $B_{m+1, m}^{T}$, which
contains only one vector, $\tilde{u}_{m+1}$


## Stability

- Accurate computation of $B_{m}^{-T} e_{m}$ can be difficult when $B_{m}^{T}$ has a large condition number
- $B_{m+1, m}^{T}=B_{m}^{T}\left[I_{m} \beta_{m+1} B_{m}^{T} e_{m}\right]$
- $\left[\begin{array}{c}-\beta_{m+1} B_{m}^{-T} e_{m} \\ 1\end{array}\right]$ is in the null space of $B_{m+1, m}^{T}$, which contains only one vector, $\tilde{u}_{m+1}$

■ Use $r_{m}^{\text {harm }}=Q_{m+1}\left[\begin{array}{c}-\beta_{m+1} B_{m}^{-T} e_{m} \\ 1\end{array}\right]=Q_{m+1} \frac{1}{\tilde{u}_{m+1, m+1}} \tilde{u}_{m+1}$

## Augmentation

■ We need to make sure we are augmenting by vectors that keep the Lanczos relations

$$
\begin{aligned}
A^{T} \hat{Q}_{k+1} & =\hat{P}_{k} \hat{B}_{k+1, k}^{T}+\alpha_{k+1} \hat{P}_{k+1} e_{k+1}^{T} \\
A \hat{P}_{k} & =\hat{Q}_{k+1} \hat{B}_{k+1, k}
\end{aligned}
$$

- $Q_{k+1}=\left[\hat{u}_{1}, \ldots, \hat{u}_{k}, r_{m}^{\text {harm }}\right]$ not an orthogonal matrix
- What should $\hat{Q}_{k+1}, \hat{P}_{k}, \hat{B}_{k+1, k}$ be to keep the above equations valid?


## Augmentation

- Compute QR-decomposition of $\left[\begin{array}{cc}g_{1}, g_{2}, \ldots, g_{k} & r_{m}^{h a r m} \\ 0\end{array}\right]$


## Augmentation

- Compute QR-decomposition of $\left[\begin{array}{cc}g_{1}, g_{2}, \ldots, g_{k} & r_{m}^{\text {harm }} \\ 0\end{array}\right]$

■ To keep a valid Lanczos Factorization:

$$
\begin{aligned}
& \hat{Q}_{k+1}=Q_{m+1} Q,(\ell \times k+1) \\
& \hat{P}_{k+1}=P_{m} \tilde{V}_{k},(n \times k) \\
& \hat{B}_{k+1, k}^{T}=\tilde{\Sigma}_{k} R_{k, k+1}^{-1}
\end{aligned}
$$

## Augmentation

- Compute QR-decomposition of $\left[\begin{array}{cc}g_{1}, g_{2}, \ldots, g_{k} & r_{m}^{\text {harm }} \\ 0\end{array}\right]$
- To keep a valid Lanczos Factorization:

$$
\begin{aligned}
& \hat{Q}_{k+1}=Q_{m+1} Q,(\ell \times k+1) \\
& \hat{P}_{k+1}=P_{m} \tilde{V}_{k},(n \times k) \\
& \hat{B}_{k+1, k}^{T}=\tilde{\Sigma}_{k} R_{k, k+1}^{-1}
\end{aligned}
$$

- Continue the Lanczos Bidiagonalization with next vector $\hat{p}_{k+1}$ and apply $m-k$ steps of the LBD:

$$
\left.\begin{array}{rl}
A^{T}\left[\hat{Q}_{k+1}\right. & \left.\hat{Q}_{m-k}\right]
\end{array}\right)=\left[\begin{array}{ll}
\hat{P}_{k} & \hat{P}_{m-k}
\end{array}\right] \hat{B}_{m+1, m}^{T}+\hat{\alpha}_{m+1} \hat{P}_{m+1} e_{m+1}^{T} .
$$

## Convergence Criteria

- The algorithm is typically restarted multiple times.
- Storage requirements are kept small.

■ Accept $\left\{\tilde{\sigma}_{j}, \hat{q}_{j}, \hat{p}_{j}\right\}$ as an approximate singular triplet:
$\sqrt{\left\|A \hat{p}_{j}-\tilde{\sigma}_{j} \hat{q}_{j}\right\|_{2}^{2}+\left\|A^{T} \hat{q}_{j}-\tilde{\sigma}_{j} \hat{p}_{j}\right\|_{2}^{2}} \leq \delta\|A\|_{2}$

1. Given $x_{0} \in \mathbb{R}^{n}$ and $\epsilon$
2. Compute $r_{0}=b-A x_{0}, \beta_{1}=\left\|r_{0}\right\|$, and $q_{1}=r_{0} / \beta_{1}$
3. Generate $Q_{m+1}, P_{m}, B_{m+1, m}$ via LBD algorithm
4. Solve $\min _{y}\left\|\beta_{1} e_{1}-B_{m+1, m} y\right\|_{2}$ using QR
5. Compute $x_{m}=x_{0}+P_{m} y$ and $r_{m}^{\text {ssqr }}=r_{0}+A P_{m} y$
6. If $\left\|A^{T} r_{m}^{\text {lsqr }}\right\|_{2} \leq \epsilon\left\|A^{T} r_{0}\right\|_{2}$ stop. Otherwise restart using $x_{0}=x_{m}$ and $r_{0}=r_{m}^{\text {sqg }}$

$$
r_{m}^{\text {sqr }}=Q_{m+1} \gamma_{m+1} Q^{\left(B_{m+1, m}\right)} e_{m+1}
$$

■ Theorem 3. The residual vector of the restarted LSQR method, $r_{m}^{\text {lsqr }}$, and the residual vector of the Augmented Harmonic Lanczos bidiagonalization method, $r_{m}^{\text {harm }}$, are multiples of each other as long as $B_{m+1, m}$ is unreduced. Moreover, $r_{m}^{\text {harm }}$ and $r_{m}^{\text {Isqr }}$ are multiples of $Q_{m+1} \tilde{u}_{m+1}$ where $\tilde{u}_{m+1}$ is the null space vector of $B_{m+1, m}^{T}$

Proof: (Baglama, Reichel, Richmond, 2012)

- Theorem 3. The residual vector of the restarted LSQR method, $r_{m}^{\text {ssqr }}$, and the residual vector of the Augmented Harmonic Lanczos bidiagonalization method, $r_{m}^{\text {harm }}$, are multiples of each other as long as $B_{m+1, m}$ is unreduced. Moreover, $r_{m}^{\text {harm }}$ and $r_{m}^{\text {Isqr }}$ are multiples of $Q_{m+1} \tilde{u}_{m+1}$ where $\tilde{u}_{m+1}$ is the null space vector of $B_{m+1, m}^{T}$

Proof: (Baglama, Reichel, Richmond, 2012)

- With the previous theorem, we can restart LSQR on the augmented space
■ By Theorem 1, doing this reduces the upper bound of the norm of the residual.

■ Compute $m$ steps of the Lanczos Bidiagonalization algorithm updating the solution and residual vectors via LSQR algorithm at each step.

- Compute Harmonic Ritz values and Harmonic Ritz vectors.
- Compute residuals of Harmonic Ritz approximations.
- If all $k$ are converged, restart and begin standard LSQR on augmented space until convergence.
else restart on augmented space and continue $m-k$ iterations of LBD and repeat.


## Rank Deficient Case

■ Linearly dependent columns/zero singular value(s)

- Non-unique solution
- Convergence to $x^{+}$as long as $x_{0}$ in range of $A^{T}$

■ Preconditioning step does not augment the Krylov subspace with vectors that approximate the null space vectors of $A$

- Approximate solution $x_{m}$ taken from an augmented Krylov subspace that is contained in the range of $A^{T}$



Example using the $1850 \times 712$ matrix ILLC1850 and the right-hand side $b$ is ILLC1850_RHS1 from the Matrix Market Collection

## E05R0000




Example using the $236 \times 236$ matrix E05R0000 and the right-hand side $b$ is E05R0000_RHS1 from the Matrix Market Collection.
Description: These matrices are from modeling 2D fluid flow in a driven cavity. The matrices are non-symmetric and indefinite. They are difficult to solve using iterative methods like preconditioned Krylov subspace methods, because it is difficult to find an effective preconditioner. The intended use of these matrices are for testing iterative solvers

## E20R0100




Example using the $4241 \times 4241$ matrix E20R0100 and the right-hand side $b$ is E20R0100_RHS1 from the Matrix Market Collection. The size of the subspace must be increased to around 140 to get good approximations, or else staggering occurs



Student Version of MATLAB
Student Version of MATLAB

Figure: Example using the $656 \times 656$ matrix CK656 and the right-hand side $b$ is a random vector.
Description: The matrix has several multiple eigenvalues and clustered eigenvalues. The eigenvalues occur in clusters of order 4; each cluster consists of two pairs of very nearly multiple eigenvalues



Example using the $468 \times 468$ matrix NOS5 and the right-hand side $b$ is a random vector.

Description: Linear Equations in structural engineering

## Conclusion/Future work

- Focus on changing the number of vectors to augment by dynamically
- Block routine
- Creating preconditioning matrix
- Applying idea to other iterative solvers
- Using Refined Harmonic Ritz Values/Vectors for approximations


## Thank You

## A Preconditioned LSQR Algorithm

James Baglama, Lothar Reichel, and Dan Richmond
Email: dan@math.uri.edu
MATLAB code alsqr.m will be available http://www.math.uri.edu/~jbaglama

