Global Residues in the Torus

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Part I

Motivation and Summary of Results

Global residue

- Let $f_1 = \cdots = f_n$ polynomial system with finitely many solutions Z_f in $\mathbb{T}^n = (\mathbb{C} - 0)^n$
- Global residue: linear function $g \mapsto \mathcal{R}_f(g)$. It provides a lot of information about the system.
- Applications in
  - Elimination
  - ideal membership
  - polynomial interpolation

Global Residue

Problem

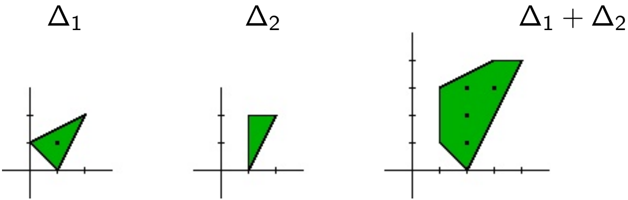
Compute the global residue as a rational function of the coefficients of the system

Sparse polynomials

Degree of f $\rightsquigarrow$ Newton polytope $\Delta(f)$

$$\Delta(f) = \text{conv.hull.}\{\text{ exponent vectors of monomials in } f\ \}$$

Example: $f_1 = a_1 x + a_2 y + a_3 x^2 y^2, \quad f_2 = b_1 x + b_2 x y^2 + b_3 x^2 y^2$



$$\# \text{ solutions } = \text{Vol}(\Delta_1, \Delta_2) = \text{Vol}(\Delta_1 + \Delta_2) - \text{Vol}(\Delta_1) - \text{Vol}(\Delta_2)$$

Summary of Results

The problem has been solved in the following cases:

Dense	$f_1, \ldots, f_n$ generic of degrees $d_1, \ldots, d_n$ [Cattani, Dickenstein, Sturmfels] '96
Unmixed	$\Delta_1, \ldots, \Delta_n$ are dilates of a single polytope [Cattani, Dickenstein] '97
Very mixed	$\Delta_1, \ldots, \Delta_n$ are in *generic relative position* [Gelfond, Khovanskii] '96
Ample	$\Delta_1, \ldots, \Delta_n$ have the same normal fan [D'Andrea, Khetan] '04
Full-dimen'l	$\Delta_1, \ldots, \Delta_n$ are arbitrary full-dimensional polytopes [S] '05

Part II

One dimensional case

$f = a_d t^d + \cdots + a_1 t + a_0$ — generic polynomial of degree d

Definition

Global residue

$$\mathcal{R}_f(g) = \sum_{f(a)=0} \operatorname{res}_a \left(\frac{g}{f} \frac{dt}{t} \right) = \sum_{f(a)=0} \frac{g(a)}{af'(a)}.$$

Note that $\mathcal{R}_f$ is

- a linear function on $\mathbb{C}[t]$
- depends rationally on the coefficients of f and g
- vanishes on $\langle f \rangle$

How to compute $\mathcal{R}_f(g)$?

By Cauchy Theorem $\mathcal{R}_f(g) = -\operatorname{res}_0\left(\frac{g}{f}\frac{dt}{t}\right) - \operatorname{res}_\infty\left(\frac{g}{f}\frac{dt}{t}\right)$.

- enough to compute for $g = t^k$, $k \in \mathbb{Z}$.
- divide g by f, $g = qf + r$ then $\mathcal{R}_f(g) = \mathcal{R}_f(r)$

k	0	1	$\cdots$	$d-1$	d	$d+1$	$d+2$	$\cdots$
$\mathcal{R}_f(t^k)$	$\frac{1}{a_0}$	0	$\cdots$	0	$\frac{1}{a_d}$	$-\frac{a_{d-1}}{a_d^2}$	$\frac{a_{d-1}^2 - a_d a_{d-2}}{a_d^3}$	$\cdots$

Note: i) $\mathcal{R}_f(g) = 0$ if $\Delta(g)$ is in the interior of $\Delta(f)$.

ii) enough to know the value of $\mathcal{R}_f$ at a single polynomial, for example, $g = tf'$

Part III

The (Toric) Euler–Jacobi Theorem

The definition of the global residue

Let $f_1, \dots, f_n$ be a collection of Laurent polynomials with finitely many common (simple) zeroes Z_f in $\mathbb{T}^n = (\mathbb{C} - 0)^n$.

Definition

Global residue

$$\mathcal{R}_f(g) = \sum_{a \in Z_f} \frac{g(a)}{J_f^{\mathbb{T}}(a)},$$

where $J_f^{\mathbb{T}}$ is the toric Jacobian $J_f^{\mathbb{T}} = \det(t_i \partial f_j / \partial t_i)$.

Note that $\mathcal{R}_f$ is

- a linear function on $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$
- depends rationally on the coefficients of the f_i and g
- vanishes on $\langle f_1, \dots, f_n \rangle$

The Toric Euler–Jacobi Theorem

Let $f_1 = \cdots = f_n = 0$ be a generic sparse system with n-dimensional Newton polytopes $\Delta_1, \ldots, \Delta_n$.
Denote $\Delta = \Delta_1 + \cdots + \Delta_n$ the Minkowski sum.

Theorem (Khovanskii, '96)

- $\mathcal{R}_f(h) = 0$ *for any* h *with* $\Delta(h) \subset \operatorname{int}(\Delta)$
- *for any* $\phi : Z_f \to \mathbb{C}$ *with* $\sum \phi(a) = 0$ *there exists* h *with* $\Delta(h) \subset \operatorname{int}(\Delta)$ *such that* $\phi(a) = h(a)/J_f^{\mathbb{T}}(a)$.

[Proof uses cohomology computations on complete intersections in smooth toric varieties.]

Part IV

How to compute $\mathcal{R}_f$?

How to compute $\mathcal{R}_f$?

Idea:

Q: Given g, can we reduce it modulo the ideal and stuff in the interior of Δ to a polynomial whose residue we know (e.g. $J_f^{\mathbb{T}}$)? i.e., can we write

$$g = h_0 + \sum_{i=1}^{n} h_i f_i + c J_f^{\mathbb{T}}, \quad \text{for } \Delta(h_0) \subset \operatorname{int}(\Delta) \text{ and } c \in \mathbb{C}?$$

A: Yes! [$\Leftarrow$ Toric Euler–Jacobi Theorem]

Q: Can we control the Newton polytopes of the h_i?

A: Yes! [$\Leftarrow$ Codimension Theorem]

Codimension Theorem

Let $\Delta_0, \dots, \Delta_n$ be n-dimensional lattice polytopes in $\mathbb{R}^n$ and Δ_0 contains the origin in its interior. Put $\tilde{\Delta} = \Delta_0 + \dots + \Delta_n$ and $\tilde{\Delta}_{(i)} = \Delta_0 + \dots + \widehat{\Delta}_i + \dots + \Delta_n$.

Theorem (S)

For generic polynomials $f_1, \dots, f_n$ with $\Delta(f_i) = \Delta_i$, the linear map

$$\bigoplus_{i=0}^{n} S_{\mathrm{int}\,(\tilde{\Delta}_{(i)})} \oplus \mathbb{C} \to S_{\mathrm{int}\,(\tilde{\Delta})}, \ (h_0, \dots, h_n, c) \mapsto h_0 + \sum_{i=1}^{n} h_i f_i + c J_f^{\mathbb{T}}$$

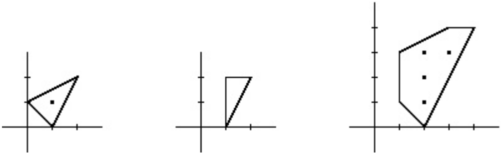
is surjective.

Here $S_{\mathrm{int}\,(\Delta)}$ denotes the space of all polynomials f with $\Delta(f)$ in the interior of Δ.

Example

$$f_1 = a_1 x + a_2 y + a_3 x^2 y^2, \quad f_2 = b_1 x + b_2 x y^2 + b_3 x^2 y^2$$

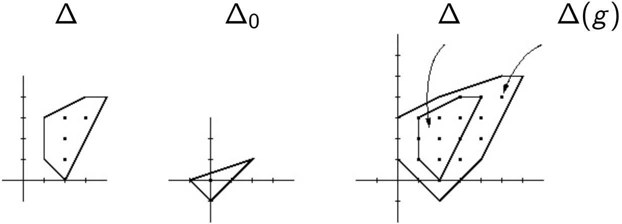
Δ_1 Δ_2 $\Delta = \Delta_1 + \Delta_2$



Compute the global residue of $g = x^5 y^4$.

Example. Step 1

Choose Δ_0 so $\tilde{\Delta} = \Delta_0 + \Delta$ contains both Δ and $\Delta(g) = (5,4)$

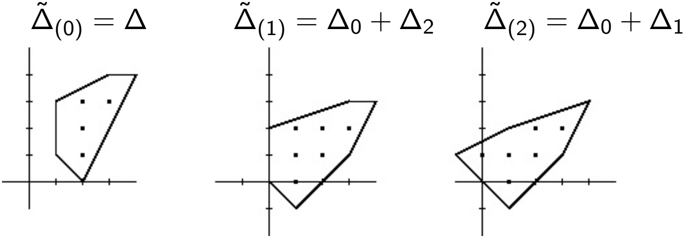


$$S_{\mathrm{int}\,(\tilde{\Delta})} = \langle xy, xy^2, xy^3, x^2, x^2y, x^2y^2, x^2y^3, x^3y, x^3y^2, x^3y^3, x^3y^4, x^4y^2, x^4y^3, x^4y^4, x^5y^4\rangle$$

$$\dim S_{\mathrm{int}\,(\tilde{\Delta})} = 15$$

Example. Step 2

Solve the linear system $g = h_0 + h_1 f_1 + h_2 f_2 + c J_f^{\mathbb{T}}$.
Here $J_f^{\mathbb{T}} = -a_1 b_1 xy - a_2 b_2 xy^3 + 2(a_1 b_3 - a_3 b_1)\, x^3 y^2 + 2 a_3 b_2 x^3 y^4$,
and the supports of h_0, h_1, and h_2 are:



$\dim S_{\text{int}\,(\tilde{\Delta}_{(0)})} = 4$, $\dim S_{\text{int}\,(\tilde{\Delta}_{(1)})} = 6$, $\dim S_{\text{int}\,(\tilde{\Delta}_{(2)})} = 6$.

Example. Answer

By the Codimension Theorem the 15 × 17 linear system

$$g = h_0 + h_1 f_1 + h_2 f_2 + c J_f^{\mathbb{T}}$$

is non-degenerate. Solving for c we get

$$c = \frac{1}{4} \frac{{a_1}^2 \, b_2}{a_3 \, (a_1 \, b_3 - a_3 \, b_1)^2}.$$

By the Toric Euler–Jacobi

$$\mathcal{R}_f(g) = \mathcal{R}_f(c J_f^{\mathbb{T}}) = c|Z_f| = \frac{{a_1}^2 \, b_2}{a_3 \, (a_1 \, b_3 - a_3 \, b_1)^2}.$$

Part VI

Open Questions

Open Questions

Q: What if the polytopes are not *n*-dimensional?

A: The Codimension Theorem fails in general. Although it remains true in some (not full-dimensional) cases those are hard to describe.

Q: What is the optimal way to choose Δ_0?

A: I don't know.

Q: Toric Euler–Jacobi implies

$$\#\text{int}\,(\Delta_1 + \cdots + \Delta_n) \cap \mathbb{Z}^n \geq \text{Vol}(\Delta_1, \ldots, \Delta_n) - 1.$$

Can we prove this geometrically?

A: I think so.