Math 411–Spring 2006

SOLUTIONS TO EXAM 2

1. Find the order of $(15, 21)$ in $\mathbb{Z}_{20} \times \mathbb{Z}_{30}$.

   The order of 15 in $\mathbb{Z}_{20}$ is 4 and the order of 21 in $\mathbb{Z}_{30}$ is 10. Therefore the order of $(15, 21)$ in $\mathbb{Z}_{20} \times \mathbb{Z}_{30}$ is $lcm(4, 10) = 20$.

2. Let H be a normal subgroup of G. Prove that the order of aH in G/H equals the smallest positive integer k such that $a^k \in H$.

   By definition the order of aH in G/H equals the smallest positive integer k such that $(aH)^k = H$. We have $(aH)^k = H$ iff $a^k H = H$ iff $a^k \in H$. Therefore the order of aH is the smallest positive integer k such that $a^k \in H$.

3. Let H be the subgroup in $\mathrm{GL}(2, \mathbb{Z}_2)$ generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. List the elements of each left coset of H. What is the index of H in G?

   Since
   $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
   the subgroup H has order 2. Thus its index is 6/2=3 (remember that $\mathrm{GL}(2, \mathbb{Z}_2)$ consists of 6 matrices with entries in $\mathbb{Z}_2$ and non-zero determinant). The cosets are

   (a) $H = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$,

   (b) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} H = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$,

   (c) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} H = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$.

4. Let $\phi : \mathbb{Z}_9 \to \mathbb{Z}_6$ be the unique homomorphism such that $\phi(2) = 2$. List the elements of $\mathrm{Ker}(\phi)$ and $\mathrm{Im}(\phi)$.

   The element $2 \in \mathbb{Z}_9$ generates the group. Thus we can find the images of all the other elements by taking multiples of 2. We get $\phi(0) = 0$, $\phi(2) = 2$, $\phi(4) = 4$, $\phi(6) = 6 = 0$, $\phi(8) = 2$, $\phi(10) = \phi(1) = 4$, $\phi(3) = 6 = 0$, $\phi(5) = 2$, $\phi(7) = 4$. From this we conclude $\mathrm{Ker}(\phi) = \{0, 3, 6\}$ and $\mathrm{Im}(\phi) = \{0, 2, 4\}$.

5. Let H and K be subgroups in S_5 generated by $(123)(45)$ and (132), respectively. List the elements of $H \cap K$.

   The order of $(123)(45)$ is 6, so it generates a group of order 6. Here are its elements:

   e,

   $(123)(45)$,

   $((123)(45))^2 = (123)^2(45)^2 = (132)$,

   $((123)(45))^3 = (132)(123)(45) = (45)$,

   $((123)(45))^4 = (132)^2 = (123)$,

   $((123)(45))^5 = (132)(45)$.

   The order of (132) is 3 and the group it generates is $\{e, (132), (123)\}$. Now the intersection is $\{e, (132), (123)\}$.

   (You can also notice that since H contains $((123)(45))^2 = (132)$ it must also contain the group generated by (132), hence, $K \subset H$. Therefore $H \cap K = K = \{e, (132), (123)\}$.)

6. How many different homomorphisms are there from $\mathbb{Z}_6$ to D_5? For each homomorphism find the image of $3 \in \mathbb{Z}_6$.

   Let $\phi : \mathbb{Z}_6 \to D_5$ be a homomorphism. You know that the order of $\mathrm{Im}(\phi)$ must divide the orders of both groups, so $|\mathrm{Im}(\phi)|$ is ether 1 or 2. In the first case ϕ is trivial (and so $\phi(3) = e$). In the second case the image is a subgroup generated by an element of order 2. Since there are 5 such elements (all reflections) we have five non-trivial homomorphisms: $\phi_i(m) = (\mu\rho^i)^m$, for $0 \le i \le 4$. In particular, $\phi_i(3) = \mu\rho^i$, for $0 \le i \le 4$.

7. (a) State the First Isomorphism Theorem.

   Let $\phi : G \to G'$ be a homomorphism. Then $G/\operatorname{Ker}(\phi) \cong \operatorname{Im}(\phi)$.

   (b) As you know $S = \{z \in \mathbb{C} \mid |z| = 1\}$ is a subgroup of the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Prove that S is isomorphic to the quotient group $\mathbb{R}/\mathbb{Z}$, where $\mathbb{R}$ is the additive group of real numbers. (Hint: Every real number θ defines a complex number $e^{2\pi i\theta} = \cos(2\pi\theta) + i\sin(2\pi\theta)$.)

   Define a homomorphism $\phi : \mathbb{R} \to \mathbb{C}^*$ by putting $\phi(\theta) = e^{2\pi i\theta}$. Clearly this is a homomorphism: $\phi(\theta_1 + \theta_2) = e^{2\pi i(\theta_1+\theta_2)} = e^{2\pi i\theta_1}e^{2\pi i\theta_2} = \phi(\theta_1)\phi(\theta_2)$.
   The image of ϕ is S since $|z| = 1$ iff $z = e^{2\pi i\theta}$ for some $\theta \in \mathbb{R}$.
   The kernel of ϕ is the set of all θ such that $e^{2\pi i\theta} = 1$, i.e. $\cos(2\pi\theta) = 1$ and $\sin(2\pi\theta) = 0$. The latter is equivalent to $\theta \in \mathbb{Z}$.
   Therefore, $\operatorname{Ker}(\phi) = \mathbb{Z}$, $\operatorname{Im}(\phi) = S$, so by the First Isomorphism Theorem $\mathbb{R}/\mathbb{Z} \cong S$.

8. (*Bonus.*) Let G be a group. For any $a, b \in G$ the element $aba^{-1}b^{-1}$ is called the commutator of a and b, and is denoted by $[a, b]$. Denote by $[G, G]$ the set of all commutators $[a, b]$ for $a, b \in G$, and all their products (of two or more).

   (a) Prove that $[G, G]$ is a normal subgroup of G. You do not have to show it is a subgroup. (Hint: First show that $g[a, b]g^{-1} = [gag^{-1}, gbg^{-1}]$.)

   (b) Prove that $G/[G, G]$ is Abelian.

   (a) The subgroup $[G, G]$ is normal iff for any $g \in G$ and any $h \in [G, G]$ we have $ghg^{-1} \in [G, G]$. Suppose $h = [a, b]$. Then

$$g[a, b]g^{-1} = gaba^{-1}b^{-1}g^{-1} = gag^{-1}gbg^{-1}ga^{-1}g^{-1}gb^{-1}g^{-1} =$$

$$= (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1} = [gag^{-1}, gbg^{-1}].$$

   Now if h is a product of commutators we can write $h = [a_1, b_1][a_2, b_2]\cdots[a_n, b_n]$, thus

$$ghg^{-1} = g[a_1, b_1][a_2, b_2]\cdots[a_n, b_n]g^{-1} = g[a_1, b_1]g^{-1}g[a_2, b_2]g^{-1}\cdots g[a_n, b_n]g^{-1}.$$

   By above the latter is again a product of commutators, i.e. lies in $[G, G]$.

   (b) To show $G/[G, G]$ is Abelian we need $(a[G, G])(b[G, G]) = (b[G, G])(a[G, G])$, i.e. $ab[G, G] = ba[G, G]$ for any $a, b \in G$. But the latter is equivalent to $a^{-1}b^{-1}ab[G, G] = [G, G]$, i.e. $a^{-1}b^{-1}ab \in [G, G]$, which is obviously true since $a^{-1}b^{-1}ab = [a^{-1}, b^{-1}]$.