DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MATH 411 April 26, 2006
EXAM 2

Your Name: ____________________________________

Your Section: 9:30 AM 11:15 AM (circle one)

This exam paper consists of 8 questions. It has 7 pages. Each answer must be justified. No calculators, books or notes are allowed!

1. (12)	__________	
2. (12)	__________	
3. (12)	__________	
4. (12)	__________	
5. (12)	__________	
6. (20)	__________	
7. (20)	__________	
8. (20)	__________	(*bonus*)
TOTAL (120)		

1. Find the order of $(15, 21)$ in $\mathbb{Z}_{20} \times \mathbb{Z}_{30}$.

2. Let H be a normal subgroup of G. Prove that the order of aH in G/H equals the smallest positive integer k such that $a^k \in H$.

3. Let H be the subgroup in $\mathrm{GL}(2, \mathbb{Z}_2)$ generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. List the elements of each left coset of H. What is the index of H in G?

4. Let $\phi : \mathbb{Z}_9 \to \mathbb{Z}_6$ be the unique homomorphism such that $\phi(2) = 2$. List the elements of $\text{Ker}(\phi)$ and $\text{Im}(\phi)$.

5. Let H and K be subgroups in S_5 generated by $(123)(45)$ and (132), respectively. List the elements of $H \cap K$.

6. How many different homomorphisms are there from $\mathbb{Z}_6$ to D_5? For each homomorphism find the image of $3 \in \mathbb{Z}_6$.

7. (a) State the First Isomorphism Theorem.

(b) As you know $S = \{z \in \mathbb{C} \mid |z| = 1\}$ is a subgroup of the multiplicative group $\mathbb{C} \setminus \{0\}$. Prove that S is isomorphic to the quotient group $\mathbb{R}/\mathbb{Z}$, where $\mathbb{R}$ is the additive group of real numbers. (Hint: Every real number θ defines a complex number $e^{2\pi i\theta} = \cos(2\pi\theta) + i\sin(2\pi\theta)$.)

8. (*Bonus.*) Let G be a group. For any $a, b \in G$ the element $aba^{-1}b^{-1}$ is called the commutator of a and b, and is denoted by $[a, b]$. Denote by $[G, G]$ the set of all commutators $[a, b]$ for $a, b \in G$, and all their products (of two or more).

   (a) Prove that $[G, G]$ is a normal subgroup of G. You do not have to show it is a subgroup. (Hint: First show that $g[a, b]g^{-1} = [gag^{-1}, gbg^{-1}]$.)

   (b) Prove that $G/[G, G]$ is Abelian.