DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MATH 411 March 15, 2006
EXAM 1

Your Name: _____

Your Section: _____

This exam paper consists of 8 questions. It has 7 pages. Each answer must be justified. No calculators, books or notes are allowed!

1. (10) _____
2. (10) _____
3. (20) _____
4. (20) _____
5. (20) _____
6. (10) _____
7. (10) _____
8. (10) _____ (*bonus*)

TOTAL (110)

1. Write 1 as a linear combination of 130 and 19 with integer coefficients.

2. Solve a system of congruences:

$$\begin{cases} 5x & \equiv & 3 & \mod 6 \\ 3x & \equiv & 1 & \mod 7 \end{cases}$$

3(a) Give the definition of a subgroup.

3(b) Let $\Sigma(2, \mathbb{R})$ be the set of all stochastic matrices in $GL(2, \mathbb{R})$, i.e. matrices whose row sums equal 1. Show that $\Sigma(2, \mathbb{R})$ is a subgroup of the multiplicative group $GL(2, \mathbb{R})$.

4(a) Give the definition of the order of an element in a group.

For the rest of the problem let G be a group of order 4.

4(b) Prove that G cannot contain elements of order 3. (Hint: Suppose $a \in G$ has order 3. Consider the element $b \in G$ not in $\langle a \rangle$. What can you say about ab?)

4(c) Prove that either G is a cyclic group or the order of every element in G is at most 2.

4(d) Write all possible group tables for G. For each table give an example of a group with this table.

5(a) Give the definition of a cyclic group.

In the rest of the problem we consider the group $\mathbb{Z}_{30}$ of integers modulo 30.

5(b) What elements generate the group? Explain.

5(c) How many distinct subgroups does the group have? How many of them are cyclic subgroups? Explain.

5(d) Draw the subgroup lattice of $\mathbb{Z}_{30}$.

6. Give an example of a subgroup of the multiplicative group $\mathbb{C} - \{0\}$ of order 3.

7. Prove that the multiplicative group $\mathbb{R} - \{0\}$ is not cyclic.

8. (*Bonus.*) Describe all possible groups with no non-trivial proper subgroups. Provide proof.