Math 411–Spring 2006

FINAL EXAM SOLUTIONS

1. Give an example of a group G such that

   (a) G is non-Abelian with non-trivial center.

   D_4 (or D_n for even n); $\mathrm{GL}(2, \mathbb{R})$ or $\mathrm{SL}(2, \mathbb{R})$ would also work

   (b) G is Abelian of order 12, but not cyclic.

   According to the Fundamental Theorem of Abelian Groups there are exactly two Abelian groups of order 12: $\mathbb{Z}_3 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. The first one is cyclic, since $\gcd(3, 4) = 1$, so G is the second.

2. Does there exist a homomorphism $\phi : \mathbb{Z}_{18} \to \mathbb{Z}_4 \times \mathbb{Z}_6$ such that $\phi(3) = (3, 3)$? If yes, construct an example; if no, give a reason.

   The order of 3 in $\mathbb{Z}_{18}$ is 6. The order of $(3, 3)$ in $\mathbb{Z}_4 \times \mathbb{Z}_6$ is $\mathrm{lcm}(|3|, |3|) = \mathrm{lcm}(4, 2) = 4$. We know that if $\phi : G \to H$ is a homomorphism then $|\phi(a)|$ divides $|a|$ for any $a \in G$. But 4 does not divide 6, therefore no such homomorphism exists.

3. How many non-isomorphic Abelian groups of order 180 are there? Write each such group as a product of cyclic groups.

   By the Fundamental Theorem of Abelian Groups, since $180 = 2^2 \cdot 3^2 \cdot 5$, we get 4 such groups:
   $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
   $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
   $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
   $\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$.

4. Which of the following groups are isomorphic? Give a short reason for each answer.

   (a) Q_8 (the quaternions)

   (b) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

   (c) $\{z \in \mathbb{C} \mid z^8 = 1\}$

   (d) $\mathbb{Z}_2 \times V_4$

   (f) $\mathbb{Z}_8$

   (g) D_4

   Both (c) and (f) are cyclic of order 8, thus isomorphic.

   Recall that $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (there are exactly two non-isomorphic groups of order 4 (cyclic and non-cyclic) and both these groups have order 4 and not cyclic), hence, (b) and (d) are isomorphic. Since they are not cyclic they are not isomorphic to (c) or (f).

   The remaining two groups (a) and (g) are not Abelian, so they cannot be isomorphic to any of (b)–(f).

   It remains to determine whether (a) is isomorphic to (g). For this, notice that Q_8 has only 1 element of order two (-1), but D_4 has 4 elements of order two (reflections). Therefore (a) and (g) are not isomorphic.

5. Let S_5 act on itself by conjugation.

   (a) Find the number of orbits of the action (i.e. conjugacy classes).

   Each conjugacy class is determined by the type of a permutation, so we need to count the number of different types. They correspond to ways of writing 5 as a sum of positive integers. We have: $1+1+1+1+1$, $1+1+1+2$, $1+1+3$, $1+2+2$, $1+4$, $2+3$, 5. Thus, there are 7 conjugacy classes.

   (b) Find the size of the orbit $\mathcal{O}_{(12)}$.

   The orbit $\mathcal{O}_{(12)}$ consists of all permutations of the same type as (12), i.e., of all transpositions. There are $\binom{5}{2} = 10$ transpositions.

   (c) Find the size of the centralizer subgroup $C_{(12)}$.

   Since the size of the conjugacy class equals the index of the centralizer subgroup we get
$$|C_{(12)}| = \frac{|S_5|}{[G : C_{(12)}]} = \frac{5!}{10} = 12.$$

6. Consider the action of the group $G = \mathbb{R}^*$ on the set $X = \mathbb{R}^2$ given by $r \cdot v = rv$ (scaling) for $r \in \mathbb{R}^*$ and $v \in \mathbb{R}^2$.

   (a) Show that this defines a group action.

   We have (i) $1 \cdot v = 1(v) = v$ for any $v \in \mathbb{R}^2$;
   (ii) $r_1 \cdot (r_2 \cdot v) = r_1 \cdot (r_2\, v) = r_1 r_2\, v = (r_1 r_2) \cdot v$ for any $r_1, r_2 \in \mathbb{R}^*$, and $v \in \mathbb{R}^2$. Thus, this is a group action.

   (b) Describe the orbits of the action. (Make sure your description agrees with the fact that X is partitioned into orbits.)

The orbit of $v = (a, b) \neq (0, 0)$ consists of all points $rv = (ra, rb)$ for $r \neq 0$, i.e., all points on the straight line determined by the vector (a, b), except for the origin. The orbit of the origin $(0, 0)$ is the origin itself since $(r \cdot 0, r \cdot 0) = (0, 0)$. Notice that no two such punctured lines intersect and their union together with the origin give a partition of the plane $\mathbb{R}^2$.

7. How many essentially different colorings of the faces of the regular tetrahedron are there if

   (a) 4 colors are available and no two faces are allowed to have the same color?

   When the tetrahedron is fixed there are 4! ways to color its faces. Indeed, the first face can be any of 4 colors, the second face can be any of the remaining 3 colors, etc. Notice that any non-trivial rotation of the tetrahedron does not preserve any of the colorings, whereas the trivial rotation preserves all of them. Therefore, the number of essentially different colorings is $\frac{1}{|A_4|}(4!) = \frac{1}{12}(24) = 2$.

   (b) 4 colors are available and faces are allowed to have the same color?

Now if we allow repeating colors we get 4^4 colorings of the faces (each face can be any of 4 colors), so we need to see which colorings are fixed by the elements of A_4, the rotational symmetries of the tetrahedron. The identity fixes all of them. Each 3-cycle fixes exactly 4^2 colorings. Indeed, the three faces containing the fixed vertex must be of the same color (4 choices) and the opposite face can be any of 4 colors. Each flip (product of two cycles) also fixes 4^2 colorings, since this time two pairs of faces must be of the same color (4 choices for each pair). Since there are 8 3-cycles and 3 flips, by Burnside's lemma the total number of essentially different colorings is:

$$\frac{1}{|A_4|}(4^4 + 8 \cdot 4^2 + 3 \cdot 4^2) = \frac{4^2}{12}(4^2 + 11) = 36.$$

8. (*Bonus.*) Let G be a group of order pq, where p and q are distinct primes. Prove that the center $Z(G)$ cannot be a proper non-trivial subgroup of G.

The quickest way to prove this is to consider the quotient group $G/Z(G)$. If we suppose that $Z(G)$ is a proper non-trivial subgroup then its order is either p or q and so the order of $G/Z(G)$ is either q or p. We know that every group of prime order is cyclic, so $G/Z(G)$ must be cyclic. But in homework you proved that if $G/Z(G)$ is cyclic then G is Abelian and so $Z(G) = G$, a contradiction.

It can also be proved using the class equation, as many of you tried. Indeed, suppose $|Z(G)| = q$ and $p < q$. Then the centralizer of every $x \notin Z(G)$ has more than q elements. Since its order divides the order of G, it must have exactly pq elements, i.e., coincide with G, which contradicts the fact that $x \notin Z(G)$.

Now suppose $|Z(G)| = p$ and $p < q$, and so $Z(G)$ is cyclic of order p. As before, the centralizer C_x of every $x \notin Z(G)$ contains more than p elements, and its order divides pq, hence it has exactly q elements! This implies that C_x is cyclic of order q. In this case, the order of every non-trivial element in C_x is q. But we know that $Z(G) \subset C_x$, and the order of every non-trivial element in $Z(G)$ is p. This is a contradiction.