## Solutions to old Exam 2 problems

## Hi students!

I am putting this version of my review for the second midterm review (place and time TBA) here on the website. DO NOT PRINT!!; it is very long!! Enjoy!!
Your course chair, Bill

PS. There are probably errors in some of the solutions presented here and for a few problems you need to complete them or simplify the answers; some questions are left to you the student. Also you might need to add more detailed explanations or justifications on the actual similar problems on your exam. I will keep updating these solutions with better corrected/improved versions. The first 5 slides are from Exam 1 practice problems but the material falls on our Exam 2.
After our exam, I will place the solutions to it right after this slide.

## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

Step 1: Find the critical points.

- Calculate $\nabla f(x, y)$ and solve


## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

## Step 1: Find the critical points.

- Calculate $\nabla f(x, y)$ and solve

$$
\nabla f(x, y)=\left\langle 2 x y-2 x, x^{2}-2 y-2\right\rangle=\langle 0,0\rangle
$$

## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

Step 1: Find the critical points.

- Calculate $\nabla f(x, y)$ and solve

$$
\nabla f(x, y)=\left\langle 2 x y-2 x, x^{2}-2 y-2\right\rangle=\langle 0,0\rangle
$$

- The first equation $2 x y-2 x=2 x(y-1)=0$ implies $x=0$ or $y=1$


## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

Step 1: Find the critical points.

- Calculate $\nabla f(x, y)$ and solve

$$
\nabla f(x, y)=\left\langle 2 x y-2 x, x^{2}-2 y-2\right\rangle=\langle 0,0\rangle
$$

- The first equation $2 x y-2 x=2 x(y-1)=0$ implies $x=0$ or $y=1$
- If $x=0$, the second equation $-2 y-2=0 \Rightarrow y=-1$.


## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

## Step 1: Find the critical points.

- Calculate $\nabla f(x, y)$ and solve

$$
\nabla f(x, y)=\left\langle 2 x y-2 x, x^{2}-2 y-2\right\rangle=\langle 0,0\rangle
$$

- The first equation $2 x y-2 x=2 x(y-1)=0$ implies $x=0$ or $y=1$
- If $x=0$, the second equation $-2 y-2=0 \Rightarrow y=-1$.
- If $y=1$, the second equation $x^{2}-4=0 \Rightarrow x= \pm 2$.


## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

## Step 1: Find the critical points.

- Calculate $\nabla f(x, y)$ and solve

$$
\nabla f(x, y)=\left\langle 2 x y-2 x, x^{2}-2 y-2\right\rangle=\langle 0,0\rangle
$$

- The first equation $2 x y-2 x=2 x(y-1)=0$ implies $x=0$ or $y=1$
- If $x=0$, the second equation $-2 y-2=0 \Rightarrow y=-1$.
- If $y=1$, the second equation $x^{2}-4=0 \Rightarrow x= \pm 2$.
- This gives a set of three critical points:

$$
\{(0,-1),(-2,1),(2,1)\} .
$$

## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

Solution: Continuation of problem 1(a).

- The set of critical points is $\{(0,-1),(-2,1),(2,1)\}$.


## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

Solution: Continuation of problem 1(a).

- The set of critical points is $\{(0,-1),(-2,1),(2,1)\}$.
- Now write the Hessian of $f(x, y)$ :


## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

## Solution: Continuation of problem 1(a).

- The set of critical points is $\{(0,-1),(-2,1),(2,1)\}$.
- Now write the Hessian of $f(x, y)$ :

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|
$$

## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

## Solution: Continuation of problem 1(a).

- The set of critical points is $\{(0,-1),(-2,1),(2,1)\}$.
- Now write the Hessian of $f(x, y)$ :

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
2 y-2 & 2 x \\
2 x & -2
\end{array}\right|
$$

## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

## Solution: Continuation of problem 1(a).

- The set of critical points is $\{(0,-1),(-2,1),(2,1)\}$.
- Now write the Hessian of $f(x, y)$ :

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
2 y-2 & 2 x \\
2 x & -2
\end{array}\right|=-4 y+4-4 x^{2}
$$

## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

## Solution: Continuation of problem 1(a).

- The set of critical points is $\{(0,-1),(-2,1),(2,1)\}$.
- Now write the Hessian of $f(x, y)$ :

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
2 y-2 & 2 x \\
2 x & -2
\end{array}\right|=-4 y+4-4 x^{2}
$$

- Apply the Second Derivative Test.
- $D(0,-1)=8>0$ and $f_{x x}=-4<0$, so $(0,-1)$ is a local maximum.


## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

## Solution: Continuation of problem 1(a).

- The set of critical points is $\{(0,-1),(-2,1),(2,1)\}$.
- Now write the Hessian of $f(x, y)$ :

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
2 y-2 & 2 x \\
2 x & -2
\end{array}\right|=-4 y+4-4 x^{2}
$$

- Apply the Second Derivative Test.
- $D(0,-1)=8>0$ and $f_{x x}=-4<0$, so $(0,-1)$ is a local maximum.
- $D(-2,1)=-16<0$, so $(-2,1)$ is a saddle point.


## Problem 1(a) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find all of the critical points of $f$ and classify them as either local maximum, local minimum, or saddle points.

## Solution: Continuation of problem 1(a).

- The set of critical points is $\{(0,-1),(-2,1),(2,1)\}$.
- Now write the Hessian of $f(x, y)$ :

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
2 y-2 & 2 x \\
2 x & -2
\end{array}\right|=-4 y+4-4 x^{2}
$$

- Apply the Second Derivative Test.
- $D(0,-1)=8>0$ and $f_{x x}=-4<0$, so $(0,-1)$ is a local maximum.
- $D(-2,1)=-16<0$, so $(-2,1)$ is a saddle point.
- $D(2,1)=-16<0$, so $(2,1)$ is a saddle point.


## Problem 1(b) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find the linearization $\mathrm{L}(x, y)$ of $f$ at the point $(1,2)$ and use it to approximate $f(0.9,2.1)$.

## Problem 1(b) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find the linearization $\mathrm{L}(x, y)$ of $f$ at the point $(1,2)$ and use it to approximate $f(0.9,2.1)$.

## Solution:

- Calculate the partial derivatives of $f$ at $(1,2)$ :


## Problem 1(b) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find the linearization $\mathrm{L}(x, y)$ of $f$ at the point $(1,2)$ and use it to approximate $f(0.9,2.1)$.

## Solution:

- Calculate the partial derivatives of $f$ at $(1,2)$ :

$$
\nabla f(x, y)=\left\langle 2 x y-2 x, x^{2}-2 y-2\right\rangle \quad \nabla f(1,2)=\langle 2,-5\rangle
$$

## Problem 1(b) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find the linearization $\mathrm{L}(x, y)$ of $f$ at the point $(1,2)$ and use it to approximate $f(0.9,2.1)$.

## Solution:

- Calculate the partial derivatives of $f$ at $(1,2)$ :

$$
\nabla f(x, y)=\left\langle 2 x y-2 x, x^{2}-2 y-2\right\rangle \quad \nabla f(1,2)=\langle 2,-5\rangle
$$

- Compute the linearization $\mathbf{L}(x, y)$ of $f$ at $(1,2)$ :


## Problem 1(b) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find the linearization $\mathrm{L}(x, y)$ of $f$ at the point $(1,2)$ and use it to approximate $f(0.9,2.1)$.

## Solution:

- Calculate the partial derivatives of $f$ at $(1,2)$ :

$$
\nabla f(x, y)=\left\langle 2 x y-2 x, x^{2}-2 y-2\right\rangle \quad \nabla f(1,2)=\langle 2,-5\rangle
$$

- Compute the linearization $\mathbf{L}(x, y)$ of $f$ at $(1,2)$ :

$$
\begin{aligned}
\mathrm{L}(x, y) & =f(1,2)+f_{x}(1,2)(x-1)+f_{y}(1,2)(y-2) \\
& =-7+2(x-1)-5(y-2)
\end{aligned}
$$

## Problem 1(b) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find the linearization $\mathrm{L}(x, y)$ of $f$ at the point $(1,2)$ and use it to approximate $f(0.9,2.1)$.

## Solution:

- Calculate the partial derivatives of $f$ at $(1,2)$ :

$$
\nabla f(x, y)=\left\langle 2 x y-2 x, x^{2}-2 y-2\right\rangle \quad \nabla f(1,2)=\langle 2,-5\rangle
$$

- Compute the linearization $\mathrm{L}(x, y)$ of $f$ at $(1,2)$ :

$$
\begin{aligned}
\mathbf{L}(x, y) & =f(1,2)+f_{x}(1,2)(x-1)+f_{y}(1,2)(y-2) \\
& =-7+2(x-1)-5(y-2)
\end{aligned}
$$

- Approximate $f(0.9,2.1)$ by $\mathbf{L}(0.9,2.1)$ :


## Problem 1(b) - Spring 2009

Let $f(x, y)=x^{2} y-y^{2}-2 y-x^{2}$.
Find the linearization $\mathbf{L}(x, y)$ of $f$ at the point $(1,2)$ and use it to approximate $f(0.9,2.1)$.

## Solution:

- Calculate the partial derivatives of $f$ at $(1,2)$ :

$$
\nabla f(x, y)=\left\langle 2 x y-2 x, x^{2}-2 y-2\right\rangle \quad \nabla f(1,2)=\langle 2,-5\rangle
$$

- Compute the linearization $\mathbf{L}(x, y)$ of $f$ at $(1,2)$ :

$$
\begin{aligned}
\mathrm{L}(x, y) & =f(1,2)+f_{x}(1,2)(x-1)+f_{y}(1,2)(y-2) \\
& =-7+2(x-1)-5(y-2)
\end{aligned}
$$

- Approximate $f(0.9,2.1)$ by $\mathbf{L}(0.9,2.1)$ :

$$
\mathrm{L}(0.9,2.1)=-7+2(-0.1)-5(0.1)=-7.7
$$

Problem 2 (a-c) - Spring 2009
Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$. (a) Find the gradient $\nabla f(x, y)$.

Problem 2 (a-c) - Spring 2009
Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$. (a) Find the gradient $\nabla f(x, y)$.

- $\nabla f(x, y)=\langle 2 x-2 y,-2 x+3+2 y\rangle$.

Problem 2 (a-c) - Spring 2009
Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$. (a) Find the gradient $\nabla f(x, y)$.

- $\nabla f(x, y)=\langle 2 x-2 y,-2 x+3+2 y\rangle$.
(b)

Find the directional derivative of $f$ at the point $(1,0)$ in the direction $\langle 3,4\rangle$.

## Problem 2 (a-c) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$. (a) Find the gradient $\nabla f(x, y)$.

- $\nabla f(x, y)=\langle 2 x-2 y,-2 x+3+2 y\rangle$.
(b)

Find the directional derivative of $f$ at the point $(1,0)$ in the direction $\langle 3,4\rangle$.

- Normalize the direction: $\mathbf{u}=\frac{\langle 3,4\rangle}{|\langle 3,4\rangle|}=\frac{1}{5}\langle 3,4\rangle$


## Problem 2 (a-c) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$. (a) Find the gradient $\nabla f(x, y)$.

- $\nabla f(x, y)=\langle 2 x-2 y,-2 x+3+2 y\rangle$.


## (b)

Find the directional derivative of $f$ at the point $(1,0)$ in the direction $\langle 3,4\rangle$.

- Normalize the direction: $\mathbf{u}=\frac{\langle 3,4\rangle}{|\langle 3,4\rangle|}=\frac{1}{5}\langle 3,4\rangle$
- Evaluate: $D_{u} f(1,0)=\nabla f(1,0) \cdot \mathbf{u}$


## Problem 2 (a-c) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$. (a) Find the gradient $\nabla f(x, y)$.

- $\nabla f(x, y)=\langle 2 x-2 y,-2 x+3+2 y\rangle$.


## (b)

Find the directional derivative of $f$ at the point $(1,0)$ in the direction $\langle 3,4\rangle$.

- Normalize the direction: $\mathbf{u}=\frac{\langle 3,4\rangle}{|\langle 3,4\rangle|}=\frac{1}{5}\langle 3,4\rangle$
- Evaluate: $D_{u} f(1,0)=\nabla f(1,0) \cdot \mathbf{u}=\langle 2,1\rangle \cdot \frac{1}{5}\langle 3,4\rangle$


## Problem 2 (a-c) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$. (a) Find the gradient $\nabla f(x, y)$.

- $\nabla f(x, y)=\langle 2 x-2 y,-2 x+3+2 y\rangle$.


## (b)

Find the directional derivative of $f$ at the point $(1,0)$ in the direction $\langle 3,4\rangle$.

- Normalize the direction: $\mathbf{u}=\frac{\langle 3,4\rangle}{|\langle 3,4\rangle|}=\frac{1}{5}\langle 3,4\rangle$
- Evaluate: $D_{u} f(1,0)=\nabla f(1,0) \cdot \mathbf{u}=\langle 2,1\rangle \cdot \frac{1}{5}\langle 3,4\rangle=2$.


## Problem 2 (a-c) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$. (a) Find the gradient $\nabla f(x, y)$.

- $\nabla f(x, y)=\langle 2 x-2 y,-2 x+3+2 y\rangle$.


## (b)

Find the directional derivative of $f$ at the point $(1,0)$ in the direction $\langle 3,4\rangle$.

- Normalize the direction: $\mathbf{u}=\frac{\langle 3,4\rangle}{|\langle 3,4\rangle|}=\frac{1}{5}\langle 3,4\rangle$
- Evaluate: $D_{u} f(1,0)=\nabla f(1,0) \cdot \mathbf{u}=\langle 2,1\rangle \cdot \frac{1}{5}\langle 3,4\rangle=2$.


## (c)

Compute all second partial derivatives of $f$.

## Problem 2 (a-c) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$. (a) Find the gradient $\nabla f(x, y)$.

- $\nabla f(x, y)=\langle 2 x-2 y,-2 x+3+2 y\rangle$.


## (b)

Find the directional derivative of $f$ at the point $(1,0)$ in the direction $\langle 3,4\rangle$.

- Normalize the direction: $\mathbf{u}=\frac{\langle 3,4\rangle}{\mid\langle 3,4\rangle}=\frac{1}{5}\langle 3,4\rangle$
- Evaluate: $D_{u} f(1,0)=\nabla f(1,0) \cdot \mathbf{u}=\langle 2,1\rangle \cdot \frac{1}{5}\langle 3,4\rangle=2$.


## (c)

Compute all second partial derivatives of $f$.

- $f_{x x}(x, y)=\frac{\partial}{\partial x}(2 x-2 y)=2$
- $f_{x y}(x, y)=f_{y x}(x, y)=\frac{\partial}{\partial y}(2 x-2 y)=-2$
- $f_{y y}(x, y)=\frac{\partial}{\partial y}(-2 x+3+2 y)=2$.


## Problem 2(d) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$. Suppose $x=s t^{2}$ and $y=e^{s-t}$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at $s=2$ and $t=1$.

## Problem 2(d) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$.
Suppose $x=s t^{2}$ and $y=e^{s-t}$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at $s=2$ and $t=1$.

## Solution:

- If $s=2$ and $t=1$, then $x=2 \cdot 1^{2}=2$ and $y=e^{2-1}=e$.


## Problem 2(d) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$.
Suppose $x=s t^{2}$ and $y=e^{s-t}$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at $s=2$ and $t=1$.

## Solution:

- If $s=2$ and $t=1$, then $x=2 \cdot 1^{2}=2$ and $y=e^{2-1}=e$.
- The Chain Rule states that

$$
\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial s}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \frac{\partial f}{\partial t}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial t}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial t} .
$$

## Problem 2(d) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$.
Suppose $x=s t^{2}$ and $y=e^{s-t}$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at $s=2$ and $t=1$.

## Solution:

- If $s=2$ and $t=1$, then $x=2 \cdot 1^{2}=2$ and $y=e^{2-1}=e$.
- The Chain Rule states that

$$
\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial s}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \frac{\partial f}{\partial t}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial t}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial t}
$$

- So, $\quad \frac{\partial f}{\partial s}=(2 x-2 y)\left(t^{2}\right)+(-2 x+3+2 y)\left(e^{s-t}\right)$


## Problem 2(d) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$.
Suppose $x=s t^{2}$ and $y=e^{s-t}$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at $s=2$ and $t=1$.

## Solution:

- If $s=2$ and $t=1$, then $x=2 \cdot 1^{2}=2$ and $y=e^{2-1}=e$.
- The Chain Rule states that

$$
\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial s}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \frac{\partial f}{\partial t}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial t}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial t}
$$

- So,

$$
\begin{aligned}
& \frac{\partial f}{\partial s}=(2 x-2 y)\left(t^{2}\right)+(-2 x+3+2 y)\left(e^{s-t}\right) \\
& =(4-2 e)+(-4+3+2 e)(e)
\end{aligned}
$$

## Problem 2(d) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$.
Suppose $x=s t^{2}$ and $y=e^{s-t}$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at $s=2$ and $t=1$.

## Solution:

- If $s=2$ and $t=1$, then $x=2 \cdot 1^{2}=2$ and $y=e^{2-1}=e$.
- The Chain Rule states that

$$
\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial s}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \frac{\partial f}{\partial t}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial t}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial t}
$$

- So,

$$
\begin{aligned}
& \frac{\partial f}{\partial s}=(2 x-2 y)\left(t^{2}\right)+(-2 x+3+2 y)\left(e^{s-t}\right) \\
= & (4-2 e)+(-4+3+2 e)(e)=4-3 e+2 e^{2}
\end{aligned}
$$

## Problem 2(d) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$.
Suppose $x=s t^{2}$ and $y=e^{s-t}$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at $s=2$ and $t=1$.

## Solution:

- If $s=2$ and $t=1$, then $x=2 \cdot 1^{2}=2$ and $y=e^{2-1}=e$.
- The Chain Rule states that

$$
\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial s}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \frac{\partial f}{\partial t}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial t}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial t}
$$

- So,

$$
\begin{aligned}
& \frac{\partial f}{\partial s}=(2 x-2 y)\left(t^{2}\right)+(-2 x+3+2 y)\left(e^{s-t}\right) \\
= & (4-2 e)+(-4+3+2 e)(e)=4-3 e+2 e^{2}
\end{aligned}
$$

- 

$$
\frac{\partial f}{\partial t}=(2 x-2 y)(2 s t)+(-2 x+3+2 y)\left(-e^{s-t}\right)
$$

## Problem 2(d) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$.
Suppose $x=s t^{2}$ and $y=e^{s-t}$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at $s=2$ and $t=1$.

## Solution:

- If $s=2$ and $t=1$, then $x=2 \cdot 1^{2}=2$ and $y=e^{2-1}=e$.
- The Chain Rule states that

$$
\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial s}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \frac{\partial f}{\partial t}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial t}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial t}
$$

- So, $\quad \frac{\partial f}{\partial s}=(2 x-2 y)\left(t^{2}\right)+(-2 x+3+2 y)\left(e^{s-t}\right)$

$$
=(4-2 e)+(-4+3+2 e)(e)=4-3 e+2 e^{2} .
$$

$\bullet$

$$
\begin{aligned}
& \frac{\partial f}{\partial t}=(2 x-2 y)(2 s t)+(-2 x+3+2 y)\left(-e^{s-t}\right) \\
= & (4-2 e)(4)+(-4+3+2 e)(-e)
\end{aligned}
$$

## Problem 2(d) - Spring 2009

Consider the function $f(x, y)=x^{2}-2 x y+3 y+y^{2}$.
Suppose $x=s t^{2}$ and $y=e^{s-t}$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at $s=2$ and $t=1$.

## Solution:

- If $s=2$ and $t=1$, then $x=2 \cdot 1^{2}=2$ and $y=e^{2-1}=e$.
- The Chain Rule states that

$$
\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial s}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \frac{\partial f}{\partial t}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial t}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial t}
$$

- So,

$$
\begin{aligned}
& \frac{\partial f}{\partial s}=(2 x-2 y)\left(t^{2}\right)+(-2 x+3+2 y)\left(e^{s-t}\right) \\
= & (4-2 e)+(-4+3+2 e)(e)=4-3 e+2 e^{2}
\end{aligned}
$$

- 

$$
\begin{gathered}
\frac{\partial f}{\partial t}=(2 x-2 y)(2 s t)+(-2 x+3+2 y)\left(-e^{s-t}\right) \\
=(4-2 e)(4)+(-4+3+2 e)(-e)=16-7 e-2 e^{2} .
\end{gathered}
$$

## Problem 3(a) - Spring 2009

Consider the function $f(x, y)=e^{x y}$ over the region $\mathbf{D}$ given by $x^{2}+4 y^{2} \leq 2$. Find the critical points of $f$.

## Problem 3(a) - Spring 2009

Consider the function $f(x, y)=e^{x y}$ over the region $\mathbf{D}$ given by $x^{2}+4 y^{2} \leq 2$. Find the critical points of $f$.

## Solution:

- $\nabla f(x, y)=\left\langle y e^{x y}, x e^{x y}\right\rangle=\langle 0,0\rangle$


## Problem 3(a) - Spring 2009

Consider the function $f(x, y)=e^{x y}$ over the region $\mathbf{D}$ given by $x^{2}+4 y^{2} \leq 2$. Find the critical points of $f$.

## Solution:

- $\nabla f(x, y)=\left\langle y e^{x y}, x e^{x y}\right\rangle=\langle 0,0\rangle$
- Since $e^{x y}$ is positive, the only critical point is $(0,0)$.

Problem 3(b) - Spring 2009
Find the extreme values on the boundary of $\mathbf{D}$.

## Problem 3(b) - Spring 2009

Find the extreme values on the boundary of $\mathbf{D}$.

## Solution:

- Use Lagrange Multipliers to study the behavior of $f$ on the boundary $g(x, y)=x^{2}+4 y^{2}=2$.


## Problem 3(b) - Spring 2009

Find the extreme values on the boundary of $D$.

## Solution:

- Use Lagrange Multipliers to study the behavior of $f$ on the boundary $g(x, y)=x^{2}+4 y^{2}=2$.

$$
\nabla f(x, y)=\left\langle y e^{x y}, x e^{x y}\right\rangle
$$

## Problem 3(b) - Spring 2009

Find the extreme values on the boundary of $D$.

## Solution:

- Use Lagrange Multipliers to study the behavior of $f$ on the boundary $g(x, y)=x^{2}+4 y^{2}=2$.

$$
\nabla f(x, y)=\left\langle y e^{x y}, x e^{x y}\right\rangle=\lambda \nabla g(x, y)
$$

## Problem 3(b) - Spring 2009

Find the extreme values on the boundary of $D$.

## Solution:

- Use Lagrange Multipliers to study the behavior of $f$ on the boundary $g(x, y)=x^{2}+4 y^{2}=2$.

$$
\begin{equation*}
\nabla f(x, y)=\left\langle y e^{x y}, x e^{x y}\right\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle \tag{1}
\end{equation*}
$$

## Problem 3(b) - Spring 2009

Find the extreme values on the boundary of $\mathbf{D}$.

## Solution:

- Use Lagrange Multipliers to study the behavior of $f$ on the boundary $g(x, y)=x^{2}+4 y^{2}=2$.

$$
\begin{equation*}
\nabla f(x, y)=\left\langle y e^{x y}, x e^{x y}\right\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle \tag{1}
\end{equation*}
$$

- Since $g(0,0) \neq 2, x=0 \Rightarrow y \neq 0$,


## Problem 3(b) - Spring 2009

Find the extreme values on the boundary of $D$.

## Solution:

- Use Lagrange Multipliers to study the behavior of $f$ on the boundary $g(x, y)=x^{2}+4 y^{2}=2$.

$$
\begin{equation*}
\nabla f(x, y)=\left\langle y e^{x y}, x e^{x y}\right\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle \tag{1}
\end{equation*}
$$

- Since $g(0,0) \neq 2, x=0 \Rightarrow y \neq 0$, and $y=0 \Rightarrow x \neq 0$.


## Problem 3(b) - Spring 2009

Find the extreme values on the boundary of $D$.

## Solution:

- Use Lagrange Multipliers to study the behavior of $f$ on the boundary $g(x, y)=x^{2}+4 y^{2}=2$.

$$
\begin{equation*}
\nabla f(x, y)=\left\langle y e^{x y}, x e^{x y}\right\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle \tag{1}
\end{equation*}
$$

- Since $g(0,0) \neq 2, x=0 \Rightarrow y \neq 0$, and $y=0 \Rightarrow x \neq 0$.
- Since $e^{x y}$ is positive and not both $x$ and $y$ are 0 , the neither is 0 by equation 1.


## Problem 3(b) - Spring 2009

Find the extreme values on the boundary of $D$.

## Solution:

- Use Lagrange Multipliers to study the behavior of $f$ on the boundary $g(x, y)=x^{2}+4 y^{2}=2$.

$$
\begin{equation*}
\nabla f(x, y)=\left\langle y e^{x y}, x e^{x y}\right\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle \tag{1}
\end{equation*}
$$

- Since $g(0,0) \neq 2, x=0 \Rightarrow y \neq 0$, and $y=0 \Rightarrow x \neq 0$.
- Since $e^{x y}$ is positive and not both $x$ and $y$ are 0 , the neither is 0 by equation 1. Hence, $\lambda / e^{x y}=y / 2 x$


## Problem 3(b) - Spring 2009

Find the extreme values on the boundary of $D$.

## Solution:

- Use Lagrange Multipliers to study the behavior of $f$ on the boundary $g(x, y)=x^{2}+4 y^{2}=2$.

$$
\begin{equation*}
\nabla f(x, y)=\left\langle y e^{x y}, x e^{x y}\right\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle \tag{1}
\end{equation*}
$$

- Since $g(0,0) \neq 2, x=0 \Rightarrow y \neq 0$, and $y=0 \Rightarrow x \neq 0$.
- Since $e^{x y}$ is positive and not both $x$ and $y$ are 0 , the neither is 0 by equation 1 . Hence, $\lambda / e^{x y}=y / 2 x=x / 8 y$


## Problem 3(b) - Spring 2009

Find the extreme values on the boundary of $D$.

## Solution:

- Use Lagrange Multipliers to study the behavior of $f$ on the boundary $g(x, y)=x^{2}+4 y^{2}=2$.

$$
\begin{equation*}
\nabla f(x, y)=\left\langle y e^{x y}, x e^{x y}\right\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle \tag{1}
\end{equation*}
$$

- Since $g(0,0) \neq 2, x=0 \Rightarrow y \neq 0$, and $y=0 \Rightarrow x \neq 0$.
- Since $e^{x y}$ is positive and not both $x$ and $y$ are 0 , the neither is 0 by equation 1 . Hence, $\lambda / e^{x y}=y / 2 x=x / 8 y \Rightarrow 8 y^{2}=2 x^{2}$.


## Problem 3(b) - Spring 2009

Find the extreme values on the boundary of D .

## Solution:

- Use Lagrange Multipliers to study the behavior of $f$ on the boundary $g(x, y)=x^{2}+4 y^{2}=2$.

$$
\begin{equation*}
\nabla f(x, y)=\left\langle y e^{x y}, x e^{x y}\right\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle \tag{1}
\end{equation*}
$$

- Since $g(0,0) \neq 2, x=0 \Rightarrow y \neq 0$, and $y=0 \Rightarrow x \neq 0$.
- Since $e^{x y}$ is positive and not both $x$ and $y$ are 0 , the neither is 0 by equation 1. Hence, $\lambda / e^{x y}=y / 2 x=x / 8 y \Rightarrow 8 y^{2}=2 x^{2}$.
- Substituting into $g(x, y)=2$ gives $2 x^{2}=2$ and $8 y^{2}=2$.


## Problem 3(b) - Spring 2009

Find the extreme values on the boundary of D .

## Solution:

- Use Lagrange Multipliers to study the behavior of $f$ on the boundary $g(x, y)=x^{2}+4 y^{2}=2$.

$$
\begin{equation*}
\nabla f(x, y)=\left\langle y e^{x y}, x e^{x y}\right\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle \tag{1}
\end{equation*}
$$

- Since $g(0,0) \neq 2, x=0 \Rightarrow y \neq 0$, and $y=0 \Rightarrow x \neq 0$.
- Since $e^{x y}$ is positive and not both $x$ and $y$ are 0 , the neither is 0 by equation 1 . Hence, $\lambda / e^{x y}=y / 2 x=x / 8 y \Rightarrow 8 y^{2}=2 x^{2}$.
- Substituting into $g(x, y)=2$ gives $2 x^{2}=2$ and $8 y^{2}=2$.
- There are four possible extremum points:

$$
\left\{\left(-1,-\frac{1}{2}\right),\left(-1, \frac{1}{2}\right),\left(1,-\frac{1}{2}\right),\left(1, \frac{1}{2}\right)\right\}
$$

## Problem 3(b) - Spring 2009

Find the extreme values on the boundary of $D$.

## Solution:

- Use Lagrange Multipliers to study the behavior of $f$ on the boundary $g(x, y)=x^{2}+4 y^{2}=2$.

$$
\begin{equation*}
\nabla f(x, y)=\left\langle y e^{x y}, x e^{x y}\right\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle \tag{1}
\end{equation*}
$$

- Since $g(0,0) \neq 2, x=0 \Rightarrow y \neq 0$, and $y=0 \Rightarrow x \neq 0$.
- Since $e^{x y}$ is positive and not both $x$ and $y$ are 0 , the neither is 0 by equation 1 . Hence, $\lambda / e^{x y}=y / 2 x=x / 8 y \Rightarrow 8 y^{2}=2 x^{2}$.
- Substituting into $g(x, y)=2$ gives $2 x^{2}=2$ and $8 y^{2}=2$.
- There are four possible extremum points:

$$
\left\{\left(-1,-\frac{1}{2}\right),\left(-1, \frac{1}{2}\right),\left(1,-\frac{1}{2}\right),\left(1, \frac{1}{2}\right)\right\}
$$

- So the extreme values of $f$ on the boundary of $D$ are:

Max $=f(1,1 / 2)=f(-1,-1 / 2)=\sqrt{e}$,
Min $=f(1,-1 / 2)=f(-1,1 / 2)=\frac{1}{\sqrt{e}}$.

## Problem 3(c) - Spring 2009

Consider the function $f(x, y)=e^{x y}$ over the region D given by $x^{2}+4 y^{2} \leq 2$. What is the absolute maximum value and absolute minimum value of $f(x, y)$ on $\mathbf{D}$ ?

## Problem 3(c) - Spring 2009

Consider the function $f(x, y)=e^{x y}$ over the region D given by $x^{2}+4 y^{2} \leq 2$. What is the absolute maximum value and absolute minimum value of $f(x, y)$ on $\mathbf{D}$ ?

## Solution:

- Recall that the only critical point of $f$ is $(0,0)$, and that on the boundary $\{(-1,-1 / 2),(-1,1 / 2),(1,-1 / 2),(1,1 / 2)\}$ are possible extremum points.


## Problem 3(c) - Spring 2009

Consider the function $f(x, y)=e^{x y}$ over the region D given by $x^{2}+4 y^{2} \leq 2$. What is the absolute maximum value and absolute minimum value of $f(x, y)$ on $\mathbf{D}$ ?

## Solution:

- Recall that the only critical point of $f$ is $(0,0)$, and that on the boundary $\{(-1,-1 / 2),(-1,1 / 2),(1,-1 / 2),(1,1 / 2)\}$ are possible extremum points.
- Calculate the value of $f$ at each point.


## Problem 3(c) - Spring 2009

Consider the function $f(x, y)=e^{x y}$ over the region D given by $x^{2}+4 y^{2} \leq 2$. What is the absolute maximum value and absolute minimum value of $f(x, y)$ on $\mathbf{D}$ ?

## Solution:

- Recall that the only critical point of $f$ is $(0,0)$, and that on the boundary $\{(-1,-1 / 2),(-1,1 / 2),(1,-1 / 2),(1,1 / 2)\}$ are possible extremum points.
- Calculate the value of $f$ at each point.
- $f(0,0)=1$


## Problem 3(c) - Spring 2009

Consider the function $f(x, y)=e^{x y}$ over the region D given by $x^{2}+4 y^{2} \leq 2$. What is the absolute maximum value and absolute minimum value of $f(x, y)$ on $\mathbf{D}$ ?

## Solution:

- Recall that the only critical point of $f$ is $(0,0)$, and that on the boundary $\{(-1,-1 / 2),(-1,1 / 2),(1,-1 / 2),(1,1 / 2)\}$ are possible extremum points.
- Calculate the value of $f$ at each point.
- $f(0,0)=1$
- $f(1,1 / 2)=f(-1,-1 / 2)=\sqrt{e}$


## Problem 3(c) - Spring 2009

Consider the function $f(x, y)=e^{x y}$ over the region D given by $x^{2}+4 y^{2} \leq 2$. What is the absolute maximum value and absolute minimum value of $f(x, y)$ on $\mathbf{D}$ ?

## Solution:

- Recall that the only critical point of $f$ is $(0,0)$, and that on the boundary $\{(-1,-1 / 2),(-1,1 / 2),(1,-1 / 2),(1,1 / 2)\}$ are possible extremum points.
- Calculate the value of $f$ at each point.
- $f(0,0)=1$
- $f(1,1 / 2)=f(-1,-1 / 2)=\sqrt{e}$
- $f(1,-1 / 2)=f(-1,1 / 2)=\frac{1}{\sqrt{e}}$


## Problem 3(c) - Spring 2009

Consider the function $f(x, y)=e^{x y}$ over the region D given by $x^{2}+4 y^{2} \leq 2$. What is the absolute maximum value and absolute minimum value of $f(x, y)$ on $\mathbf{D}$ ?

## Solution:

- Recall that the only critical point of $f$ is $(0,0)$, and that on the boundary $\{(-1,-1 / 2),(-1,1 / 2),(1,-1 / 2),(1,1 / 2)\}$ are possible extremum points.
- Calculate the value of $f$ at each point.
- $f(0,0)=1$
- $f(1,1 / 2)=f(-1,-1 / 2)=\sqrt{e}$
- $f(1,-1 / 2)=f(-1,1 / 2)=\frac{1}{\sqrt{e}}$
- So, the maximum value is $\sqrt{e}$ and the minimum value is $\frac{1}{\sqrt{e}}$.


## Problem 4(a) - Spring 2009

Evaluate the following iterated integral.

$$
\int_{-1}^{2} \int_{0}^{1}\left(x^{2} y-x y\right) d y d x
$$

## Problem 4(a) - Spring 2009

Evaluate the following iterated integral.

$$
\int_{-1}^{2} \int_{0}^{1}\left(x^{2} y-x y\right) d y d x
$$

## Solution:

$$
\int_{-1}^{2} \int_{0}^{1}\left(x^{2} y-x y\right) d y d x
$$

## Problem 4(a) - Spring 2009

Evaluate the following iterated integral.

$$
\int_{-1}^{2} \int_{0}^{1}\left(x^{2} y-x y\right) d y d x
$$

## Solution:

$$
\int_{-1}^{2} \int_{0}^{1}\left(x^{2} y-x y\right) d y d x=\int_{-1}^{2}\left[x^{2} \frac{y^{2}}{2}-x \frac{y^{2}}{2}\right]_{0}^{1} d x
$$

## Problem 4(a) - Spring 2009

Evaluate the following iterated integral.

$$
\int_{-1}^{2} \int_{0}^{1}\left(x^{2} y-x y\right) d y d x
$$

## Solution:

$$
\begin{gathered}
\int_{-1}^{2} \int_{0}^{1}\left(x^{2} y-x y\right) d y d x=\int_{-1}^{2}\left[x^{2} \frac{y^{2}}{2}-x \frac{y^{2}}{2}\right]_{0}^{1} d x \\
=\int_{-1}^{2}\left(\frac{x^{2}}{2}-\frac{x}{2}\right) d x
\end{gathered}
$$

## Problem 4(a) - Spring 2009

Evaluate the following iterated integral.

$$
\int_{-1}^{2} \int_{0}^{1}\left(x^{2} y-x y\right) d y d x
$$

## Solution:

$$
\begin{gathered}
\int_{-1}^{2} \int_{0}^{1}\left(x^{2} y-x y\right) d y d x=\int_{-1}^{2}\left[x^{2} \frac{y^{2}}{2}-x \frac{y^{2}}{2}\right]_{0}^{1} d x \\
=\int_{-1}^{2}\left(\frac{x^{2}}{2}-\frac{x}{2}\right) d x=\left[\frac{x^{3}}{6}-\frac{x^{2}}{4}\right]_{-1}^{2}
\end{gathered}
$$

## Problem 4(a) - Spring 2009

Evaluate the following iterated integral.

$$
\int_{-1}^{2} \int_{0}^{1}\left(x^{2} y-x y\right) d y d x
$$

## Solution:

$$
\begin{gathered}
\int_{-1}^{2} \int_{0}^{1}\left(x^{2} y-x y\right) d y d x=\int_{-1}^{2}\left[x^{2} \frac{y^{2}}{2}-x \frac{y^{2}}{2}\right]_{0}^{1} d x \\
=\int_{-1}^{2}\left(\frac{x^{2}}{2}-\frac{x}{2}\right) d x=\left[\frac{x^{3}}{6}-\frac{x^{2}}{4}\right]_{-1}^{2} \\
=\left(\frac{2^{3}}{6}-\frac{2^{2}}{4}\right)-\left(\frac{(-1)^{3}}{6}-\frac{(-1)^{2}}{4}\right)=\frac{16-12+2+3}{12}=\frac{3}{4}
\end{gathered}
$$

## Problem 4(b) - Spring 2009

Find the volume $\mathbf{V}$ of the region below $z=x^{2}-2 x y+3$ and above the rectangle $\mathbf{R}=[0,1] \times[-1,1]$.

## Problem 4(b) - Spring 2009

Find the volume $\mathbf{V}$ of the region below $z=x^{2}-2 x y+3$ and above the rectangle $\mathbf{R}=[0,1] \times[-1,1]$.

## Solution: Calculate using Fubini's Theorem.

$$
\mathbf{V}=\iint_{\mathbf{R}}\left(x^{2}-2 x y+3\right) d A=\int_{0}^{1} \int_{-1}^{1}\left(x^{2}-2 x y+3\right) d y d x
$$

## Problem 4(b) - Spring 2009

Find the volume $\mathbf{V}$ of the region below $z=x^{2}-2 x y+3$ and above the rectangle $\mathbf{R}=[0,1] \times[-1,1]$.

## Solution: Calculate using Fubini's Theorem.

$$
\begin{aligned}
& \mathbf{V}=\iint_{R}\left(x^{2}-2 x y+3\right) d A=\int_{0}^{1} \int_{-1}^{1}\left(x^{2}-2 x y+3\right) d y d x \\
& =\int_{-1}^{1} \int_{0}^{1}\left(x^{2}-2 x y+3\right) d x d y=\int_{-1}^{1}\left[\frac{x^{3}}{3}-x^{2} y+3 x\right]_{0}^{1} d y
\end{aligned}
$$

## Problem 4(b) - Spring 2009

Find the volume $\mathbf{V}$ of the region below $z=x^{2}-2 x y+3$ and above the rectangle $\mathbf{R}=[0,1] \times[-1,1]$.

## Solution: Calculate using Fubini's Theorem.

$$
\begin{aligned}
& \mathbf{V}=\iint_{R}\left(x^{2}-2 x y+3\right) d A=\int_{0}^{1} \int_{-1}^{1}\left(x^{2}-2 x y+3\right) d y d x \\
& =\int_{-1}^{1} \int_{0}^{1}\left(x^{2}-2 x y+3\right) d x d y=\int_{-1}^{1}\left[\frac{x^{3}}{3}-x^{2} y+3 x\right]_{0}^{1} d y \\
& =\int_{-1}^{1}\left(\frac{1}{3}-y+3\right) d y
\end{aligned}
$$

## Problem 4(b) - Spring 2009

Find the volume $\mathbf{V}$ of the region below $z=x^{2}-2 x y+3$ and above the rectangle $\mathbf{R}=[0,1] \times[-1,1]$.

## Solution: Calculate using Fubini's Theorem.

$$
\begin{gathered}
\mathbf{V}=\iint_{R}\left(x^{2}-2 x y+3\right) d A=\int_{0}^{1} \int_{-1}^{1}\left(x^{2}-2 x y+3\right) d y d x \\
=\int_{-1}^{1} \int_{0}^{1}\left(x^{2}-2 x y+3\right) d x d y=\int_{-1}^{1}\left[\frac{x^{3}}{3}-x^{2} y+3 x\right]_{0}^{1} d y \\
=\int_{-1}^{1}\left(\frac{1}{3}-y+3\right) d y=\left[\frac{y}{3}-\frac{y^{2}}{2}+3 y\right]_{-1}^{1}
\end{gathered}
$$

## Problem 4(b) - Spring 2009

Find the volume $\mathbf{V}$ of the region below $z=x^{2}-2 x y+3$ and above the rectangle $\mathbf{R}=[0,1] \times[-1,1]$.

Solution: Calculate using Fubini's Theorem.

$$
\begin{aligned}
& \mathbf{V}=\iint_{R}\left(x^{2}-2 x y+3\right) d A=\int_{0}^{1} \int_{-1}^{1}\left(x^{2}-2 x y+3\right) d y d x \\
& =\int_{-1}^{1} \int_{0}^{1}\left(x^{2}-2 x y+3\right) d x d y=\int_{-1}^{1}\left[\frac{x^{3}}{3}-x^{2} y+3 x\right]_{0}^{1} d y \\
& \quad=\int_{-1}^{1}\left(\frac{1}{3}-y+3\right) d y=\left[\frac{y}{3}-\frac{y^{2}}{2}+3 y\right]_{-1}^{1} \\
& =\left(\frac{1}{3}-\frac{1^{2}}{2}+3(1)\right)-\left(\frac{-1}{3}-\frac{(-1)^{2}}{2}+3(-1)\right)=6+\frac{2}{3}
\end{aligned}
$$

## Problem 5(a) - Spring 2009

Consider the surface $S$ given by the equation $x^{2}+y^{3}+z^{2}=0$. Give an equation for the tangent plane of $S$ at the point (2, -2, 2).

## Problem 5(a) - Spring 2009

Consider the surface S given by the equation $x^{2}+y^{3}+z^{2}=0$.
Give an equation for the tangent plane of $S$ at the point (2, -2, 2).

## Solution:

- Let $f(x, y, z)=x^{2}+y^{3}+z^{2}$.


## Problem 5(a) - Spring 2009

Consider the surface S given by the equation $x^{2}+y^{3}+z^{2}=0$. Give an equation for the tangent plane of $S$ at the point (2, -2, 2).

## Solution:

- Let $f(x, y, z)=x^{2}+y^{3}+z^{2}$.
- Compute the gradient of $f$ at $(2,-2,2)$.

$$
\nabla f(x, y, z)=\left\langle 2 x, 3 y^{2}, 2 z\right\rangle \quad \nabla f(2,-2,2)=\langle 4,12,4\rangle .
$$

## Problem 5(a) - Spring 2009

Consider the surface $S$ given by the equation $x^{2}+y^{3}+z^{2}=0$. Give an equation for the tangent plane of $S$ at the point (2, -2, 2).

## Solution:

- Let $f(x, y, z)=x^{2}+y^{3}+z^{2}$.
- Compute the gradient of $f$ at $(2,-2,2)$.

$$
\nabla f(x, y, z)=\left\langle 2 x, 3 y^{2}, 2 z\right\rangle \quad \nabla f(2,-2,2)=\langle 4,12,4\rangle
$$

- The equation for the tangent plane is

$$
\nabla f(2,-2,2) \cdot\langle x-2, y+2, z-2\rangle=0
$$

## Problem 5(a) - Spring 2009

Consider the surface $S$ given by the equation $x^{2}+y^{3}+z^{2}=0$. Give an equation for the tangent plane of $S$ at the point (2, -2, 2).

## Solution:

- Let $f(x, y, z)=x^{2}+y^{3}+z^{2}$.
- Compute the gradient of $f$ at $(2,-2,2)$.

$$
\nabla f(x, y, z)=\left\langle 2 x, 3 y^{2}, 2 z\right\rangle \quad \nabla f(2,-2,2)=\langle 4,12,4\rangle
$$

- The equation for the tangent plane is

$$
\begin{gather*}
\nabla f(2,-2,2) \cdot\langle x-2, y+2, z-2\rangle=0 \\
4(x-2)+12(y+2)+4(z-2)=0
\end{gather*}
$$

## Problem 5(b) - Spring 2009

Consider the surface $S$ given by the equation $x^{2}+y^{3}+z^{2}=0$. Give an equation for the normal line to $S$ at the point (2, -2, 2).

## Problem 5(b) - Spring 2009

Consider the surface $S$ given by the equation $x^{2}+y^{3}+z^{2}=0$. Give an equation for the normal line to $S$ at the point (2, -2, 2).

## Solution:

- Let $f(x, y, z)=x^{2}+y^{3}+z^{2}$.


## Problem 5(b) - Spring 2009

Consider the surface $S$ given by the equation $x^{2}+y^{3}+z^{2}=0$. Give an equation for the normal line to $S$ at the point (2, -2, 2).

## Solution:

- Let $f(x, y, z)=x^{2}+y^{3}+z^{2}$.
- Compute the gradient of $f$ at $(2,-2,2)$.

$$
\nabla f(x, y, z)=\left\langle 2 x, 3 y^{2}, 2 z\right\rangle \quad \nabla f(2,-2,2)=\langle 4,12,4\rangle
$$

## Problem 5(b) - Spring 2009

Consider the surface $S$ given by the equation $x^{2}+y^{3}+z^{2}=0$. Give an equation for the normal line to $S$ at the point (2, -2, 2).

## Solution:

- Let $f(x, y, z)=x^{2}+y^{3}+z^{2}$.
- Compute the gradient of $f$ at $(2,-2,2)$.

$$
\nabla f(x, y, z)=\left\langle 2 x, 3 y^{2}, 2 z\right\rangle \quad \nabla f(2,-2,2)=\langle 4,12,4\rangle
$$

- The equation for the normal line is

$$
r(t)=\langle 2,-2,2\rangle+t\langle 4,12,4\rangle .
$$

## Problem 26(a) - Exam 1 - Fall 2006

Let $g(x, y)=y e^{x}$. Estimate $g(0.1,1.9)$ using the linear approximation $\mathrm{L}(x, y)$ of $g(x, y)$ at $(x, y)=(0,2)$.

## Problem 26(a) - Exam 1 - Fall 2006

Let $g(x, y)=y e^{x}$. Estimate $g(0.1,1.9)$ using the linear approximation $\mathrm{L}(x, y)$ of $g(x, y)$ at $(x, y)=(0,2)$.

## Solution:

- Calculating partial derivatives at $(0,2)$, we obtain:

$$
\begin{array}{cc}
g_{x}(x, y)=y e^{x} & g_{y}(x, y)=e^{x} \\
g_{x}(0,2)=2 & g_{y}(0,2)=1
\end{array}
$$

## Problem 26(a) - Exam 1 - Fall 2006

Let $g(x, y)=y e^{x}$. Estimate $g(0.1,1.9)$ using the linear approximation $\mathrm{L}(x, y)$ of $g(x, y)$ at $(x, y)=(0,2)$.

## Solution:

- Calculating partial derivatives at $(0,2)$, we obtain:

$$
\begin{array}{cc}
g_{x}(x, y)=y e^{x} & g_{y}(x, y)=e^{x} \\
g_{x}(0,2)=2 & g_{y}(0,2)=1
\end{array}
$$

- Let $\mathbf{L}(x, y)$ be the linear approximation at $(0,2)$.


## Problem 26(a) - Exam 1 - Fall 2006

Let $g(x, y)=y e^{x}$. Estimate $g(0.1,1.9)$ using the linear approximation $\mathrm{L}(x, y)$ of $g(x, y)$ at $(x, y)=(0,2)$.

## Solution:

- Calculating partial derivatives at $(0,2)$, we obtain:

$$
\begin{array}{cc}
g_{x}(x, y)=y e^{x} & g_{y}(x, y)=e^{x} \\
g_{x}(0,2)=2 & g_{y}(0,2)=1
\end{array}
$$

- Let $\mathrm{L}(x, y)$ be the linear approximation at $(0,2)$.

$$
\mathrm{L}(x, y)=g(0,2)+g_{x}(0,2)(x-0)+g_{y}(0,2)(y-2)
$$

## Problem 26(a) - Exam 1 - Fall 2006

Let $g(x, y)=y e^{x}$. Estimate $g(0.1,1.9)$ using the linear approximation $\mathrm{L}(x, y)$ of $g(x, y)$ at $(x, y)=(0,2)$.

## Solution:

- Calculating partial derivatives at $(0,2)$, we obtain:

$$
\begin{array}{cc}
g_{x}(x, y)=y e^{x} & g_{y}(x, y)=e^{x} \\
g_{x}(0,2)=2 & g_{y}(0,2)=1
\end{array}
$$

- Let $\mathrm{L}(x, y)$ be the linear approximation at $(0,2)$.

$$
\begin{gathered}
\mathbf{L}(x, y)=g(0,2)+g_{x}(0,2)(x-0)+g_{y}(0,2)(y-2) \\
\mathbf{L}(x, y)=2+2 x+(y-2) .
\end{gathered}
$$

## Problem 26(a) - Exam 1 - Fall 2006

Let $g(x, y)=y e^{x}$. Estimate $g(0.1,1.9)$ using the linear approximation $\mathrm{L}(x, y)$ of $g(x, y)$ at $(x, y)=(0,2)$.

## Solution:

- Calculating partial derivatives at $(0,2)$, we obtain:

$$
\begin{array}{cc}
g_{x}(x, y)=y e^{x} & g_{y}(x, y)=e^{x} \\
g_{x}(0,2)=2 & g_{y}(0,2)=1
\end{array}
$$

- Let $\mathrm{L}(x, y)$ be the linear approximation at $(0,2)$.

$$
\begin{gathered}
\mathrm{L}(x, y)=g(0,2)+g_{x}(0,2)(x-0)+g_{y}(0,2)(y-2) \\
\mathrm{L}(x, y)=2+2 x+(y-2) .
\end{gathered}
$$

- Calculating at (0.1, 1.9):

$$
\mathrm{L}(0.1,1.9)
$$

## Problem 26(a) - Exam 1 - Fall 2006

Let $g(x, y)=y e^{x}$. Estimate $g(0.1,1.9)$ using the linear approximation $\mathrm{L}(x, y)$ of $g(x, y)$ at $(x, y)=(0,2)$.

## Solution:

- Calculating partial derivatives at $(0,2)$, we obtain:

$$
\begin{array}{cc}
g_{x}(x, y)=y e^{x} & g_{y}(x, y)=e^{x} \\
g_{x}(0,2)=2 & g_{y}(0,2)=1
\end{array}
$$

- Let $\mathrm{L}(x, y)$ be the linear approximation at $(0,2)$.

$$
\begin{gathered}
\mathrm{L}(x, y)=g(0,2)+g_{x}(0,2)(x-0)+g_{y}(0,2)(y-2) \\
\mathrm{L}(x, y)=2+2 x+(y-2) .
\end{gathered}
$$

- Calculating at (0.1, 1.9):

$$
\mathrm{L}(0.1,1.9)=2+2(0.1)+(1.9-2)
$$

## Problem 26(a) - Exam 1 - Fall 2006

Let $g(x, y)=y e^{x}$. Estimate $g(0.1,1.9)$ using the linear approximation $\mathrm{L}(x, y)$ of $g(x, y)$ at $(x, y)=(0,2)$.

## Solution:

- Calculating partial derivatives at $(0,2)$, we obtain:

$$
\begin{array}{cc}
g_{x}(x, y)=y e^{x} & g_{y}(x, y)=e^{x} \\
g_{x}(0,2)=2 & g_{y}(0,2)=1
\end{array}
$$

- Let $\mathrm{L}(x, y)$ be the linear approximation at $(0,2)$.

$$
\begin{gathered}
\mathrm{L}(x, y)=g(0,2)+g_{x}(0,2)(x-0)+g_{y}(0,2)(y-2) \\
\mathrm{L}(x, y)=2+2 x+(y-2) .
\end{gathered}
$$

- Calculating at (0.1, 1.9):

$$
\mathrm{L}(0.1,1.9)=2+2(0.1)+(1.9-2)=2+.2-.1
$$

## Problem 26(a) - Exam 1 - Fall 2006

Let $g(x, y)=y e^{x}$. Estimate $g(0.1,1.9)$ using the linear approximation $\mathrm{L}(x, y)$ of $g(x, y)$ at $(x, y)=(0,2)$.

## Solution:

- Calculating partial derivatives at $(0,2)$, we obtain:

$$
\begin{array}{cc}
g_{x}(x, y)=y e^{x} & g_{y}(x, y)=e^{x} \\
g_{x}(0,2)=2 & g_{y}(0,2)=1
\end{array}
$$

- Let $\mathrm{L}(x, y)$ be the linear approximation at $(0,2)$.

$$
\begin{gathered}
\mathbf{L}(x, y)=g(0,2)+g_{x}(0,2)(x-0)+g_{y}(0,2)(y-2) \\
\mathbf{L}(x, y)=2+2 x+(y-2) .
\end{gathered}
$$

- Calculating at (0.1, 1.9):

$$
\mathrm{L}(0.1,1.9)=2+2(0.1)+(1.9-2)=2+.2-.1=2.1
$$

Problem 36-Exam 1
Find an equation for the tangent plane to the graph of $f(x, y)=y \ln x$ at $(1,4,0)$.

## Problem 36 - Exam 1

Find an equation for the tangent plane to the graph of $f(x, y)=y \ln x$ at $(1,4,0)$.

## Solution:

- Recall that the tangent plane to a surface $z=f(x, y)$ at the point $P=\left(x_{0}, y_{0}, z_{0}\right)$ is:

$$
z-z_{0}=f_{x}\left(x_{0}, y_{0}\right)\left(x-x_{0}\right)+f_{y}\left(x_{0}, y_{0}\right)\left(y-y_{0}\right)
$$

## Problem 36 - Exam 1

Find an equation for the tangent plane to the graph of $f(x, y)=y \ln x$ at $(1,4,0)$.

## Solution:

- Recall that the tangent plane to a surface $z=f(x, y)$ at the point $P=\left(x_{0}, y_{0}, z_{0}\right)$ is:

$$
z-z_{0}=f_{x}\left(x_{0}, y_{0}\right)\left(x-x_{0}\right)+f_{y}\left(x_{0}, y_{0}\right)\left(y-y_{0}\right)
$$

- Calculating partial derivatives, we obtain:

$$
\begin{array}{cc}
f_{x}(x, y)=\frac{y}{x} & f_{y}(x, y)=\ln x \\
f_{x}(1,4)=4 & f_{y}(1,4)=\ln 1=0
\end{array}
$$

## Problem 36 - Exam 1

Find an equation for the tangent plane to the graph of $f(x, y)=y \ln x$ at $(1,4,0)$.

## Solution:

- Recall that the tangent plane to a surface $z=f(x, y)$ at the point $P=\left(x_{0}, y_{0}, z_{0}\right)$ is:

$$
z-z_{0}=f_{x}\left(x_{0}, y_{0}\right)\left(x-x_{0}\right)+f_{y}\left(x_{0}, y_{0}\right)\left(y-y_{0}\right)
$$

- Calculating partial derivatives, we obtain:

$$
\begin{array}{cc}
f_{x}(x, y)=\frac{y}{x} & f_{y}(x, y)=\ln x \\
f_{x}(1,4)=4 & f_{y}(1,4)=\ln 1=0
\end{array}
$$

- The equation of the tangent plane is:

$$
z=4(x-1)+0 \cdot(y-4)=4 x-4
$$

## Problem 40 - Exam 1

Explain why the limit of $f(x, y)=\left(3 x^{2} y^{2}\right) /\left(2 x^{4}+y^{4}\right)$ does not exist as $(x, y)$ approaches $(0,0)$.

## Problem 40 - Exam 1

Explain why the limit of $f(x, y)=\left(3 x^{2} y^{2}\right) /\left(2 x^{4}+y^{4}\right)$ does not exist as $(x, y)$ approaches $(0,0)$.

## Solution:

- Along the line $\langle t, t\rangle, t \neq 0, f(x, y)$ has the constant value $\frac{3}{3}=1$.


## Problem 40 - Exam 1

Explain why the limit of $f(x, y)=\left(3 x^{2} y^{2}\right) /\left(2 x^{4}+y^{4}\right)$ does not exist as $(x, y)$ approaches $(0,0)$.

## Solution:

- Along the line $\langle t, t\rangle, t \neq 0, f(x, y)$ has the constant value $\frac{3}{3}=1$.
- Along the line $\langle 0, t\rangle, t \neq 0, f(x, y)$ has the constant value $\frac{0}{1}=0$.


## Problem 40 - Exam 1

Explain why the limit of $f(x, y)=\left(3 x^{2} y^{2}\right) /\left(2 x^{4}+y^{4}\right)$ does not exist as $(x, y)$ approaches $(0,0)$.

## Solution:

- Along the line $\langle t, t\rangle, t \neq 0, f(x, y)$ has the constant value $\frac{3}{3}=1$.
- Along the line $\langle 0, t\rangle, t \neq 0, f(x, y)$ has the constant value $\frac{0}{1}=0$.
- Since $f(x, y)$ has 2 different limiting values at $(0,0)$, it does not have a limit at $(0,0)$.

Problem 42(a) - Exam 1
Find all of the first order and second order partial derivatives of the function $f(x, y)=x^{3}-x y^{2}+y$.

## Problem 42(a) - Exam 1

Find all of the first order and second order partial derivatives of the function $f(x, y)=x^{3}-x y^{2}+y$.

## Solution:

- First calculate the first order partial derivatives:

$$
f_{x}(x, y)=3 x^{2}-y^{2} \quad f_{y}(x, y)=-2 x y+1
$$

## Problem 42(a) - Exam 1

Find all of the first order and second order partial derivatives of the function $f(x, y)=x^{3}-x y^{2}+y$.

## Solution:

- First calculate the first order partial derivatives:

$$
f_{x}(x, y)=3 x^{2}-y^{2} \quad f_{y}(x, y)=-2 x y+1
$$

- The second order partial derivatives $f_{x x}, f_{x y}, f_{y x}$ and $f_{y y}$ are:

$$
\begin{array}{cc}
f_{x x}(x, y)=6 x & f_{x y}(x, y)=-2 y \\
f_{y x}(x, y)=-2 y & f_{y y}(x, y)=-2 x
\end{array}
$$

## Problem 43 - Exam 1

Find the linear approximation $\mathrm{L}(x, y)$ of the function $f(x, y)=x y e^{x}$ at $(x, y)=(1,1)$, and use it to estimate $f(1.1,0.9)$.

## Problem 43 - Exam 1

Find the linear approximation $\mathrm{L}(x, y)$ of the function $f(x, y)=x y e^{x}$ at $(x, y)=(1,1)$, and use it to estimate $f(1.1,0.9)$.

## Solution:

- Calculating partial derivatives at $(1,1)$, we obtain:

$$
\begin{gathered}
f_{x}(x, y)=y e^{x}+x y e^{x} \quad f_{y}(x, y)=x e^{x} \\
f_{x}(1,1)=2 e \quad f_{y}(1,1)=e .
\end{gathered}
$$

## Problem 43 - Exam 1

Find the linear approximation $\mathrm{L}(x, y)$ of the function $f(x, y)=x y e^{x}$ at $(x, y)=(1,1)$, and use it to estimate $f(1.1,0.9)$.

## Solution:

- Calculating partial derivatives at $(1,1)$, we obtain:

$$
\begin{gathered}
f_{x}(x, y)=y e^{x}+x y e^{x} \quad f_{y}(x, y)=x e^{x} \\
f_{x}(1,1)=2 e \quad f_{y}(1,1)=e .
\end{gathered}
$$

- Let $\mathbf{L}(x, y)$ be the linear approximation at $(1,1)$.


## Problem 43 - Exam 1

Find the linear approximation $\mathrm{L}(x, y)$ of the function $f(x, y)=x y e^{x}$ at $(x, y)=(1,1)$, and use it to estimate $f(1.1,0.9)$.

## Solution:

- Calculating partial derivatives at $(1,1)$, we obtain:

$$
\begin{gathered}
f_{x}(x, y)=y e^{x}+x y e^{x} \quad f_{y}(x, y)=x e^{x} \\
f_{x}(1,1)=2 e \quad f_{y}(1,1)=e .
\end{gathered}
$$

- Let $\mathbf{L}(x, y)$ be the linear approximation at $(1,1)$.

$$
\mathrm{L}(x, y)=f(1,1)+f_{x}(1,1)(x-1)+f_{y}(1,1)(y-1)
$$

## Problem 43 - Exam 1

Find the linear approximation $\mathrm{L}(x, y)$ of the function $f(x, y)=x y e^{x}$ at $(x, y)=(1,1)$, and use it to estimate $f(1.1,0.9)$.

## Solution:

- Calculating partial derivatives at $(1,1)$, we obtain:

$$
\begin{gathered}
f_{x}(x, y)=y e^{x}+x y e^{x} \quad f_{y}(x, y)=x e^{x} \\
f_{x}(1,1)=2 e \quad f_{y}(1,1)=e .
\end{gathered}
$$

- Let $\mathbf{L}(x, y)$ be the linear approximation at $(1,1)$.

$$
\begin{gathered}
\mathrm{L}(x, y)=f(1,1)+f_{x}(1,1)(x-1)+f_{y}(1,1)(y-1) \\
\mathbf{L}(x, y)=e+2 e(x-1)+e(y-1) .
\end{gathered}
$$

## Problem 43 - Exam 1

Find the linear approximation $\mathrm{L}(x, y)$ of the function $f(x, y)=x y e^{x}$ at $(x, y)=(1,1)$, and use it to estimate $f(1.1,0.9)$.

## Solution:

- Calculating partial derivatives at $(1,1)$, we obtain:

$$
\begin{gathered}
f_{x}(x, y)=y e^{x}+x y e^{x} \quad f_{y}(x, y)=x e^{x} \\
f_{x}(1,1)=2 e \quad f_{y}(1,1)=e .
\end{gathered}
$$

- Let $\mathbf{L}(x, y)$ be the linear approximation at $(1,1)$.

$$
\begin{gathered}
\mathrm{L}(x, y)=f(1,1)+f_{x}(1,1)(x-1)+f_{y}(1,1)(y-1) \\
\mathbf{L}(x, y)=e+2 e(x-1)+e(y-1) .
\end{gathered}
$$

- Calculating at $(1.1,0.9)$, we obtain:

$$
\mathrm{L}(1.1,0.9)
$$

## Problem 43 - Exam 1

Find the linear approximation $\mathrm{L}(x, y)$ of the function $f(x, y)=x y e^{x}$ at $(x, y)=(1,1)$, and use it to estimate $f(1.1,0.9)$.

## Solution:

- Calculating partial derivatives at $(1,1)$, we obtain:

$$
\begin{gathered}
f_{x}(x, y)=y e^{x}+x y e^{x} \quad f_{y}(x, y)=x e^{x} \\
f_{x}(1,1)=2 e \quad f_{y}(1,1)=e .
\end{gathered}
$$

- Let $\mathbf{L}(x, y)$ be the linear approximation at $(1,1)$.

$$
\begin{gathered}
\mathrm{L}(x, y)=f(1,1)+f_{x}(1,1)(x-1)+f_{y}(1,1)(y-1) \\
\mathbf{L}(x, y)=e+2 e(x-1)+e(y-1) .
\end{gathered}
$$

- Calculating at $(1.1,0.9)$, we obtain:

$$
\mathbf{L}(1.1,0.9)=e+2 e(0.1)+e(-0.1)
$$

## Problem 43 - Exam 1

Find the linear approximation $\mathrm{L}(x, y)$ of the function $f(x, y)=x y e^{x}$ at $(x, y)=(1,1)$, and use it to estimate $f(1.1,0.9)$.

## Solution:

- Calculating partial derivatives at $(1,1)$, we obtain:

$$
\begin{gathered}
f_{x}(x, y)=y e^{x}+x y e^{x} \quad f_{y}(x, y)=x e^{x} \\
f_{x}(1,1)=2 e \quad f_{y}(1,1)=e .
\end{gathered}
$$

- Let $\mathbf{L}(x, y)$ be the linear approximation at $(1,1)$.

$$
\begin{gathered}
\mathrm{L}(x, y)=f(1,1)+f_{x}(1,1)(x-1)+f_{y}(1,1)(y-1) \\
\mathbf{L}(x, y)=e+2 e(x-1)+e(y-1) .
\end{gathered}
$$

- Calculating at $(1.1,0.9)$, we obtain:

$$
\mathrm{L}(1.1,0.9)=e+2 e(0.1)+e(-0.1)=1.1 e
$$

Problem 1 - Exam 2 - Fall 2008
(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$


## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ?


## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ?
$\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle$


## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ?
$\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle$.


## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ? $\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle$.
(3) Calculate the directional derivative of $f$ at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 3,4\rangle$ ?


## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ? $\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle$.
(3) Calculate the directional derivative of $f$ at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 3,4\rangle$ ?
- $\mathbf{u}=\frac{v}{|v|}=\frac{1}{5}\langle 3,4\rangle$ is the unit vector in the direction of $\langle 3,4\rangle$.


## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ?
$\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle$.
(3) Calculate the directional derivative of $f$ at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 3,4\rangle$ ?
- $\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle$ is the unit vector in the direction of $\langle 3,4\rangle$.
- Next evaluate

$$
D_{\mathbf{u}} f(1,2)=\nabla f(1,2) \cdot \mathbf{u}
$$

## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ? $\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle$.
(3) Calculate the directional derivative of $f$ at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 3,4\rangle$ ?
- $\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle$ is the unit vector in the direction of $\langle 3,4\rangle$.
- Next evaluate

$$
D_{\mathbf{u}} f(1,2)=\nabla f(1,2) \cdot \mathbf{u}=\langle 8,4\rangle \cdot \frac{1}{5}\langle 3,4\rangle
$$

## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ? $\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle$.
(3) Calculate the directional derivative of $f$ at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 3,4\rangle$ ?
- $\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle$ is the unit vector in the direction of $\langle 3,4\rangle$.
- Next evaluate

$$
D_{\mathbf{u}} f(1,2)=\nabla f(1,2) \cdot \mathbf{u}=\langle 8,4\rangle \cdot \frac{1}{5}\langle 3,4\rangle=\frac{1}{5}(24+16)
$$

## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ? $\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle$.
(3) Calculate the directional derivative of $f$ at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 3,4\rangle$ ?
- $\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle$ is the unit vector in the direction of $\langle 3,4\rangle$.
- Next evaluate

$$
D_{\mathrm{u}} f(1,2)=\nabla f(1,2) \cdot \mathbf{u}=\langle 8,4\rangle \cdot \frac{1}{5}\langle 3,4\rangle=\frac{1}{5}(24+16)=8 .
$$

## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ?

$$
\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle
$$

(3) Calculate the directional derivative of $f$ at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 3,4\rangle$ ?

- $\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle$ is the unit vector in the direction of $\langle 3,4\rangle$.
- Next evaluate

$$
D_{\mathrm{u}} f(1,2)=\nabla f(1,2) \cdot \mathbf{u}=\langle 8,4\rangle \cdot \frac{1}{5}\langle 3,4\rangle=\frac{1}{5}(24+16)=8 .
$$

(9) What is the linearization $\mathbf{L}(x, y)$ of $f$ at $(1,2)$ ?

## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ? $\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle$.
(3) Calculate the directional derivative of $f$ at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 3,4\rangle$ ?
- $\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle$ is the unit vector in the direction of $\langle 3,4\rangle$.
- Next evaluate

$$
D_{\mathrm{u}} f(1,2)=\nabla f(1,2) \cdot \mathbf{u}=\langle 8,4\rangle \cdot \frac{1}{5}\langle 3,4\rangle=\frac{1}{5}(24+16)=8 .
$$

(9) What is the linearization $\mathbf{L}(x, y)$ of $f$ at $(1,2)$ ?

$$
\mathrm{L}(x, y)=f(1,2)+f_{x}(1,2)(x-1)+f_{y}(1,2)(y-2)
$$

## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ?

$$
\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle
$$

(3) Calculate the directional derivative of $f$ at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 3,4\rangle$ ?

- $\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle$ is the unit vector in the direction of $\langle 3,4\rangle$.
- Next evaluate

$$
D_{\mathrm{u}} f(1,2)=\nabla f(1,2) \cdot \mathbf{u}=\langle 8,4\rangle \cdot \frac{1}{5}\langle 3,4\rangle=\frac{1}{5}(24+16)=8 .
$$

(9) What is the linearization $\mathbf{L}(x, y)$ of $f$ at $(1,2)$ ?

$$
\begin{aligned}
\mathrm{L}(x, y) & =f(1,2)+f_{x}(1,2)(x-1)+f_{y}(1,2)(y-2) \\
& =6+8(x-1)+4(x-2)
\end{aligned}
$$

## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ?

$$
\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle
$$

(3) Calculate the directional derivative of $f$ at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 3,4\rangle$ ?

- $\mathbf{u}=\frac{\mathbf{v}}{|v|}=\frac{1}{5}\langle 3,4\rangle$ is the unit vector in the direction of $\langle 3,4\rangle$.
- Next evaluate

$$
D_{\mathrm{u}} f(1,2)=\nabla f(1,2) \cdot \mathbf{u}=\langle 8,4\rangle \cdot \frac{1}{5}\langle 3,4\rangle=\frac{1}{5}(24+16)=8 .
$$

(9) What is the linearization $\mathbf{L}(x, y)$ of $f$ at $(1,2)$ ?

$$
\begin{aligned}
\mathrm{L}(x, y) & =f(1,2)+f_{x}(1,2)(x-1)+f_{y}(1,2)(y-2) \\
& =6+8(x-1)+4(x-2)
\end{aligned}
$$

(6) Use the linearization $\mathrm{L}(x, y)$ in the previous part to estimate $f(0.9,2.1)$.

## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ?

$$
\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle
$$

(3) Calculate the directional derivative of $f$ at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 3,4\rangle$ ?

- $\mathbf{u}=\frac{\mathbf{v}}{|v|}=\frac{1}{5}\langle 3,4\rangle$ is the unit vector in the direction of $\langle 3,4\rangle$.
- Next evaluate

$$
D_{\mathrm{u}} f(1,2)=\nabla f(1,2) \cdot \mathbf{u}=\langle 8,4\rangle \cdot \frac{1}{5}\langle 3,4\rangle=\frac{1}{5}(24+16)=8 .
$$

(9) What is the linearization $\mathbf{L}(x, y)$ of $f$ at $(1,2)$ ?

$$
\begin{aligned}
\mathrm{L}(x, y) & =f(1,2)+f_{x}(1,2)(x-1)+f_{y}(1,2)(y-2) \\
& =6+8(x-1)+4(x-2)
\end{aligned}
$$

(6) Use the linearization $\mathrm{L}(x, y)$ in the previous part to estimate $f(0.9,2.1)$.
$\mathrm{L}(0.9,2.1)$

## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ?

$$
\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle
$$

(3) Calculate the directional derivative of $f$ at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 3,4\rangle$ ?

- $\mathbf{u}=\frac{\mathbf{v}}{|v|}=\frac{1}{5}\langle 3,4\rangle$ is the unit vector in the direction of $\langle 3,4\rangle$.
- Next evaluate

$$
D_{\mathrm{u}} f(1,2)=\nabla f(1,2) \cdot \mathbf{u}=\langle 8,4\rangle \cdot \frac{1}{5}\langle 3,4\rangle=\frac{1}{5}(24+16)=8 .
$$

(9) What is the linearization $\mathbf{L}(x, y)$ of $f$ at $(1,2)$ ?

$$
\begin{aligned}
\mathrm{L}(x, y) & =f(1,2)+f_{x}(1,2)(x-1)+f_{y}(1,2)(y-2) \\
& =6+8(x-1)+4(x-2)
\end{aligned}
$$

(6) Use the linearization $\mathrm{L}(x, y)$ in the previous part to estimate $f(0.9,2.1)$.
$L(0.9,2.1)=6+8(0.9-1)+4(2.1-2)$

## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ?

$$
\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle
$$

(3) Calculate the directional derivative of $f$ at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 3,4\rangle$ ?

- $\mathbf{u}=\frac{\mathbf{v}}{|\boldsymbol{v}|}=\frac{1}{5}\langle 3,4\rangle$ is the unit vector in the direction of $\langle 3,4\rangle$.
- Next evaluate

$$
D_{\mathrm{u}} f(1,2)=\nabla f(1,2) \cdot \mathbf{u}=\langle 8,4\rangle \cdot \frac{1}{5}\langle 3,4\rangle=\frac{1}{5}(24+16)=8 .
$$

(9) What is the linearization $\mathbf{L}(x, y)$ of $f$ at $(1,2)$ ?

$$
\begin{aligned}
\mathrm{L}(x, y) & =f(1,2)+f_{x}(1,2)(x-1)+f_{y}(1,2)(y-2) \\
& =6+8(x-1)+4(x-2)
\end{aligned}
$$

(6) Use the linearization $\mathrm{L}(x, y)$ in the previous part to estimate $f(0.9,2.1)$.
$\mathrm{L}(0.9,2.1)=6+8(0.9-1)+4(2.1-2)=6-.8+.4$

## Problem 1 - Exam 2 - Fall 2008

(1) For the function $f(x, y)=2 x^{2}+x y^{2}$, calculate $f_{x}, f_{y}, f_{x y}, f_{x x}$ :

- $f_{x}(x, y)=4 x+y^{2}$
- $f_{y}(x, y)=2 x y$
- $f_{x y}(x, y)=2 y$
- $f_{x x}(x, y)=4$
(2) What is the gradient $\nabla f(x, y)$ of $f$ at the point $(1,2)$ ?

$$
\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\left\langle 4 x+y^{2}, 2 x y\right\rangle \quad \nabla f(1,2)=\langle 8,4\rangle
$$

(3) Calculate the directional derivative of $f$ at the point $(1,2)$ in the direction of the vector $\mathbf{v}=\langle 3,4\rangle$ ?

- $\mathbf{u}=\frac{\mathbf{v}}{|\boldsymbol{v}|}=\frac{1}{5}\langle 3,4\rangle$ is the unit vector in the direction of $\langle 3,4\rangle$.
- Next evaluate

$$
D_{\mathrm{u}} f(1,2)=\nabla f(1,2) \cdot \mathbf{u}=\langle 8,4\rangle \cdot \frac{1}{5}\langle 3,4\rangle=\frac{1}{5}(24+16)=8 .
$$

(9) What is the linearization $\mathrm{L}(x, y)$ of $f$ at $(1,2)$ ?

$$
\begin{aligned}
\mathrm{L}(x, y) & =f(1,2)+f_{x}(1,2)(x-1)+f_{y}(1,2)(y-2) \\
& =6+8(x-1)+4(x-2)
\end{aligned}
$$

(6) Use the linearization $\mathrm{L}(x, y)$ in the previous part to estimate $f(0.9,2.1)$.
$\mathrm{L}(0.9,2.1)=6+8(0.9-1)+4(2.1-2)=6-.8+.4=5.6$

## Problem 2(a) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P(2,-1,79)$ heading North, is she ascending or descending? Justify your answers.

## Problem 2(a) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P(2,-1,79)$ heading North, is she ascending or descending? Justify your answers.

## Solution:

- Let $f(x, y)=z=100-4 x^{2}-5 y^{2}$.


## Problem 2(a) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P(2,-1,79)$ heading North, is she ascending or descending? Justify your answers.

## Solution:

- Let $f(x, y)=z=100-4 x^{2}-5 y^{2}$.
- This is a problem where we need to calculate the sign of the directional derivative $D_{\langle 0,1\rangle} f(2,-1)=\nabla f(2,-1) \cdot\langle 0,1\rangle$, where $\langle 0,1\rangle$ represents North.


## Problem 2(a) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P(2,-1,79)$ heading North, is she ascending or descending? Justify your answers.

## Solution:

- Let $f(x, y)=z=100-4 x^{2}-5 y^{2}$.
- This is a problem where we need to calculate the sign of the directional derivative $D_{\langle 0,1\rangle} f(2,-1)=\nabla f(2,-1) \cdot\langle 0,1\rangle$, where $\langle 0,1\rangle$ represents North.
- Calculating, we obtain:

$$
\nabla f(x, y)=\langle-8 x,-10 y\rangle \quad \nabla f(2,-1)=\langle-16,10\rangle .
$$

## Problem 2(a) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P(2,-1,79)$ heading North, is she ascending or descending? Justify your answers.

## Solution:

- Let $f(x, y)=z=100-4 x^{2}-5 y^{2}$.
- This is a problem where we need to calculate the sign of the directional derivative $D_{\langle 0,1\rangle} f(2,-1)=\nabla f(2,-1) \cdot\langle 0,1\rangle$, where $\langle 0,1\rangle$ represents North.
- Calculating, we obtain:

$$
\nabla f(x, y)=\langle-8 x,-10 y\rangle \quad \nabla f(2,-1)=\langle-16,10\rangle
$$

- Hence,

$$
D_{\langle 0,1\rangle} f(2,-1)
$$

## Problem 2(a) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P(2,-1,79)$ heading North, is she ascending or descending? Justify your answers.

## Solution:

- Let $f(x, y)=z=100-4 x^{2}-5 y^{2}$.
- This is a problem where we need to calculate the sign of the directional derivative $D_{\langle 0,1\rangle} f(2,-1)=\nabla f(2,-1) \cdot\langle 0,1\rangle$, where $\langle 0,1\rangle$ represents North.
- Calculating, we obtain:

$$
\nabla f(x, y)=\langle-8 x,-10 y\rangle \quad \nabla f(2,-1)=\langle-16,10\rangle .
$$

- Hence,

$$
D_{\langle 0,1\rangle} f(2,-1)=\langle-16,10\rangle \cdot\langle 0,1\rangle
$$

## Problem 2(a) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P(2,-1,79)$ heading North, is she ascending or descending? Justify your answers.

## Solution:

- Let $f(x, y)=z=100-4 x^{2}-5 y^{2}$.
- This is a problem where we need to calculate the sign of the directional derivative $D_{\langle 0,1\rangle} f(2,-1)=\nabla f(2,-1) \cdot\langle 0,1\rangle$, where $\langle 0,1\rangle$ represents North.
- Calculating, we obtain:

$$
\nabla f(x, y)=\langle-8 x,-10 y\rangle \quad \nabla f(2,-1)=\langle-16,10\rangle
$$

- Hence,

$$
D_{\langle 0,1\rangle} f(2,-1)=\langle-16,10\rangle \cdot\langle 0,1\rangle=10>0,
$$

## Problem 2(a) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P(2,-1,79)$ heading North, is she ascending or descending? Justify your answers.

## Solution:

- Let $f(x, y)=z=100-4 x^{2}-5 y^{2}$.
- This is a problem where we need to calculate the sign of the directional derivative $D_{\langle 0,1\rangle} f(2,-1)=\nabla f(2,-1) \cdot\langle 0,1\rangle$, where $\langle 0,1\rangle$ represents North.
- Calculating, we obtain:

$$
\nabla f(x, y)=\langle-8 x,-10 y\rangle \quad \nabla f(2,-1)=\langle-16,10\rangle .
$$

- Hence,

$$
D_{\langle 0,1\rangle} f(2,-1)=\langle-16,10\rangle \cdot\langle 0,1\rangle=10>0,
$$

which means that she is ascending.

## Problem 2(b) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Justify your answers.
When the hiker is at the point $Q(1,0,96)$, in which direction on the map should she initially head to descend most rapidly?

## Problem 2(b) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Justify your answers.
When the hiker is at the point $Q(1,0,96)$, in which direction on the map should she initially head to descend most rapidly?

## Solution:

- Recall that $\nabla f(x, y)=\langle-8 x,-10 y\rangle$.


## Problem 2(b) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Justify your answers.
When the hiker is at the point $Q(1,0,96)$, in which direction on the map should she initially head to descend most rapidly?

## Solution:

- Recall that $\nabla f(x, y)=\langle-8 x,-10 y\rangle$.
- The direction of greatest descent is in the direction $\mathbf{v}$ of $-\nabla f$ at the point $(1,0)$ in the $x y$-plane.


## Problem 2(b) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Justify your answers.
When the hiker is at the point $Q(1,0,96)$, in which direction on the map should she initially head to descend most rapidly?

## Solution:

- Recall that $\nabla f(x, y)=\langle-8 x,-10 y\rangle$.
- The direction of greatest descent is in the direction $\mathbf{v}$ of $-\nabla f$ at the point $(1,0)$ in the $x y$-plane.
- Thus,

$$
\mathbf{v}=-\nabla f(1,0)
$$

## Problem 2(b) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Justify your answers.
When the hiker is at the point $Q(1,0,96)$, in which direction on the map should she initially head to descend most rapidly?

## Solution:

- Recall that $\nabla f(x, y)=\langle-8 x,-10 y\rangle$.
- The direction of greatest descent is in the direction $\mathbf{v}$ of $-\nabla f$ at the point $(1,0)$ in the $x y$-plane.
- Thus,

$$
\mathbf{v}=-\nabla f(1,0)=\langle 8,0\rangle
$$

## Problem 2(b) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Justify your answers.
When the hiker is at the point $Q(1,0,96)$, in which direction on the map should she initially head to descend most rapidly?

## Solution:

- Recall that $\nabla f(x, y)=\langle-8 x,-10 y\rangle$.
- The direction of greatest descent is in the direction $\mathbf{v}$ of $-\nabla f$ at the point $(1,0)$ in the $x y$-plane.
- Thus,

$$
\mathbf{v}=-\nabla f(1,0)=\langle 8,0\rangle
$$

which means that she should go East.

## Problem 2(c) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. What is her rate of descent when she travels at a speed of 10 meters per minute in the direction of maximal decent from $Q(1,0,96)$ ? Justify your answers.

## Problem 2(c) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. What is her rate of descent when she travels at a speed of 10 meters per minute in the direction of maximal decent from $Q(1,0,96)$ ? Justify your answers.

## Solution:

- By velocity, we mean the velocity of the projection on the $x y$-plane or map (the wording is somewhat ambiguous).


## Problem 2(c) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. What is her rate of descent when she travels at a speed of 10 meters per minute in the direction of maximal decent from $Q(1,0,96)$ ? Justify your answers.

## Solution:

- By velocity, we mean the velocity of the projection on the $x y$-plane or map (the wording is somewhat ambiguous).
- By part (b), if she travels at unit speed (in measurements on the map which we don't know) in the direction of $\nabla f(1,0)$ (which is East $=\langle 1,0\rangle$ ), then her maximal rate of decent is $|\nabla f(1,0)|=8$ (in measurements of the map).


## Problem 2(c) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. What is her rate of descent when she travels at a speed of 10 meters per minute in the direction of maximal decent from $Q(1,0,96)$ ? Justify your answers.

## Solution:

- By velocity, we mean the velocity of the projection on the $x y$-plane or map (the wording is somewhat ambiguous).
- By part (b), if she travels at unit speed (in measurements on the map which we don't know) in the direction of $\nabla f(1,0)$ (which is East $=\langle 1,0\rangle$ ), then her maximal rate of decent is $|\nabla f(1,0)|=8$ (in measurements of the map).
- So, her rate of decent in the direction of greatest decent is $10 \cdot 8$ meters $/$ minute $=80$ meters $/$ minute.


## Problem 2(d) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. When the hiker is at the point $Q(1,0,96)$, in which two directions on her map can she initially head to neither ascend nor descend (to keep traveling at the same height)? Justify your answers.

## Problem 2(d) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North.
When the hiker is at the point $Q(1,0,96)$, in which two directions on her map can she initially head to neither ascend nor descend (to keep traveling at the same height)? Justify your answers.

## Solution:

- First find all the possible vectors $\mathbf{v}=\langle x, y\rangle$ which are orthogonal to $\nabla f(1,0)=\langle-8,0\rangle$ :


## Problem 2(d) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North.
When the hiker is at the point $Q(1,0,96)$, in which two directions on her map can she initially head to neither ascend nor descend (to keep traveling at the same height)? Justify your answers.

## Solution:

- First find all the possible vectors $\mathbf{v}=\langle x, y\rangle$ which are orthogonal to $\nabla f(1,0)=\langle-8,0\rangle$ :

$$
\langle-8,0\rangle \cdot\langle x, y\rangle=-8 x+0=0
$$

## Problem 2(d) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North.
When the hiker is at the point $Q(1,0,96)$, in which two directions on her map can she initially head to neither ascend nor descend (to keep traveling at the same height)? Justify your answers.

## Solution:

- First find all the possible vectors $\mathbf{v}=\langle x, y\rangle$ which are orthogonal to $\nabla f(1,0)=\langle-8,0\rangle$ :

$$
\langle-8,0\rangle \cdot\langle x, y\rangle=-8 x+0=0 \Longrightarrow x=0
$$

## Problem 2(d) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. When the hiker is at the point $Q(1,0,96)$, in which two directions on her map can she initially head to neither ascend nor descend (to keep traveling at the same height)? Justify your answers.

## Solution:

- First find all the possible vectors $\mathbf{v}=\langle x, y\rangle$ which are orthogonal to $\nabla f(1,0)=\langle-8,0\rangle$ :

$$
\langle-8,0\rangle \cdot\langle x, y\rangle=-8 x+0=0 \Longrightarrow x=0
$$

- Therefore, at the point $Q(1,0,96)$ and in the map directions of the vectors $\pm\langle 0,1\rangle$, she is neither ascending or descending.


## Problem 2(d) - Fall 2008

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=100-4 x^{2}-5 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. When the hiker is at the point $Q(1,0,96)$, in which two directions on her map can she initially head to neither ascend nor descend (to keep traveling at the same height)? Justify your answers.

## Solution:

- First find all the possible vectors $\mathbf{v}=\langle x, y\rangle$ which are orthogonal to $\nabla f(1,0)=\langle-8,0\rangle$ :

$$
\langle-8,0\rangle \cdot\langle x, y\rangle=-8 x+0=0 \Longrightarrow x=0
$$

- Therefore, at the point $Q(1,0,96)$ and in the map directions of the vectors $\pm\langle 0,1\rangle$, she is neither ascending or descending. These directions are North and South.


## Problem 3(a) - Fall 2008

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 3 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Problem 3(a) - Fall 2008

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 3 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}
$$

## Problem 3(a) - Fall 2008

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 3 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

## Problem 3(a) - Fall 2008

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 3 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

- Note that when $t=1$, then $x=1$ and $y=2$


## Problem 3(a) - Fall 2008

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 3 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

- Note that when $t=1$, then $x=1$ and $y=2$ and that $\frac{d x}{d t}=3 t^{2}$ and $\frac{d y}{d t}=2 t$


## Problem 3(a) - Fall 2008

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 3 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

- Note that when $t=1$, then $x=1$ and $y=2$ and that $\frac{d x}{d t}=3 t^{2}$ and $\frac{d y}{d t}=2 t \Longrightarrow \frac{d x}{d t}(1)=3$ and $\frac{d y}{d t}(1)=2$.


## Problem 3(a) - Fall 2008

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 3 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

- Note that when $t=1$, then $x=1$ and $y=2$ and that $\frac{d x}{d t}=3 t^{2}$ and

$$
\frac{d y}{d t}=2 t \Longrightarrow \frac{d x}{d t}(1)=3 \text { and } \frac{d y}{d t}(1)=2 .
$$

- Plug in the values in the table into the Chain Rule at $t=1$ :


## Problem 3(a) - Fall 2008

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 3 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

- Note that when $t=1$, then $x=1$ and $y=2$ and that $\frac{d x}{d t}=3 t^{2}$ and

$$
\frac{d y}{d t}=2 t \Longrightarrow \frac{d x}{d t}(1)=3 \text { and } \frac{d y}{d t}(1)=2 .
$$

- Plug in the values in the table into the Chain Rule at $t=1$ :

$$
f^{\prime}(1)=\frac{\partial f}{\partial x}(1,2) \cdot 3+\frac{\partial f}{\partial y}(1,2) \cdot 2
$$

## Problem 3(a) - Fall 2008

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 3 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

- Note that when $t=1$, then $x=1$ and $y=2$ and that $\frac{d x}{d t}=3 t^{2}$ and

$$
\frac{d y}{d t}=2 t \Longrightarrow \frac{d x}{d t}(1)=3 \text { and } \frac{d y}{d t}(1)=2 .
$$

- Plug in the values in the table into the Chain Rule at $t=1$ :

$$
f^{\prime}(1)=\frac{\partial f}{\partial x}(1,2) \cdot 3+\frac{\partial f}{\partial y}(1,2) \cdot 2=(-1) \cdot 3+3 \cdot 2
$$

## Problem 3(a) - Fall 2008

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 3 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

- Note that when $t=1$, then $x=1$ and $y=2$ and that $\frac{d x}{d t}=3 t^{2}$ and

$$
\frac{d y}{d t}=2 t \Longrightarrow \frac{d x}{d t}(1)=3 \text { and } \frac{d y}{d t}(1)=2 .
$$

- Plug in the values in the table into the Chain Rule at $t=1$ :

$$
f^{\prime}(1)=\frac{\partial f}{\partial x}(1,2) \cdot 3+\frac{\partial f}{\partial y}(1,2) \cdot 2=(-1) \cdot 3+3 \cdot 2=3 .
$$

## Problem 3(b) - Fall 2008

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v^{2}, y=u-v^{2} .
$$

## Problem 3(b) - Fall 2008

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v^{2}, y=u-v^{2}
$$

## Solution:

- By the Chain Rule we have:

$$
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v}
$$

## Problem 3(b) - Fall 2008

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v^{2}, y=u-v^{2}
$$

## Solution:

- By the Chain Rule we have:

$$
\begin{gathered}
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \\
=\left(3 x^{2} y^{2}+y^{3}\right)(2 v)+\left(2 x^{3} y+3 y^{2} x\right)(-2 v)
\end{gathered}
$$

## Problem 3(b) - Fall 2008

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v^{2}, y=u-v^{2}
$$

## Solution:

- By the Chain Rule we have:

$$
\begin{gathered}
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \\
=\left(3 x^{2} y^{2}+y^{3}\right)(2 v)+\left(2 x^{3} y+3 y^{2} x\right)(-2 v) .
\end{gathered}
$$

## Problem 3(b) - Fall 2008

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v^{2}, y=u-v^{2}
$$

## Solution:

- By the Chain Rule we have:

$$
\begin{gathered}
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \\
=\left(3 x^{2} y^{2}+y^{3}\right)(2 v)+\left(2 x^{3} y+3 y^{2} x\right)(-2 v) .
\end{gathered}
$$

- When $u=1$ and $v=1$, then $x=1^{2}+1^{2}=2$ and $y=1-1^{2}=0$.


## Problem 3(b) - Fall 2008

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v^{2}, y=u-v^{2}
$$

## Solution:

- By the Chain Rule we have:

$$
\begin{gathered}
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \\
=\left(3 x^{2} y^{2}+y^{3}\right)(2 v)+\left(2 x^{3} y+3 y^{2} x\right)(-2 v) .
\end{gathered}
$$

- When $u=1$ and $v=1$, then $x=1^{2}+1^{2}=2$ and $y=1-1^{2}=0$.
- So for $u=1$ and $v=1$, we get:


## Problem 3(b) - Fall 2008

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v^{2}, y=u-v^{2} .
$$

## Solution:

- By the Chain Rule we have:

$$
\begin{gathered}
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \\
=\left(3 x^{2} y^{2}+y^{3}\right)(2 v)+\left(2 x^{3} y+3 y^{2} x\right)(-2 v) .
\end{gathered}
$$

- When $u=1$ and $v=1$, then $x=1^{2}+1^{2}=2$ and $y=1-1^{2}=0$.
- So for $u=1$ and $v=1$, we get:

$$
\frac{\partial z}{\partial v}=0
$$

## Problem 4 - Fall 2008

Consider the surface $x^{2}+y^{2}-2 z^{2}=0$ and the point $P(1,1,1)$ which lies on the surface.
(i) Find the equation of the tangent plane to the surface at $P$.
(ii) Find the equation of the normal line to the surface at $P$.

## Problem 4 - Fall 2008

Consider the surface $x^{2}+y^{2}-2 z^{2}=0$ and the point $P(1,1,1)$ which lies on the surface.
(i) Find the equation of the tangent plane to the surface at $P$.
(ii) Find the equation of the normal line to the surface at $P$.

## Solution:

- Recall that the gradient of $\mathbf{F}(x, y, z)=x^{2}+y^{2}-2 z^{2}$ is normal $\mathbf{n}$ to the surface.


## Problem 4 - Fall 2008

Consider the surface $x^{2}+y^{2}-2 z^{2}=0$ and the point $P(1,1,1)$ which lies on the surface.
(i) Find the equation of the tangent plane to the surface at $P$.
(ii) Find the equation of the normal line to the surface at $P$.

## Solution:

- Recall that the gradient of $\mathbf{F}(x, y, z)=x^{2}+y^{2}-2 z^{2}$ is normal $\mathbf{n}$ to the surface.
- Calculating, we obtain:


## Problem 4 - Fall 2008

Consider the surface $x^{2}+y^{2}-2 z^{2}=0$ and the point $P(1,1,1)$ which lies on the surface.
(i) Find the equation of the tangent plane to the surface at $P$.
(ii) Find the equation of the normal line to the surface at $P$.

## Solution:

- Recall that the gradient of $\mathbf{F}(x, y, z)=x^{2}+y^{2}-2 z^{2}$ is normal $\mathbf{n}$ to the surface.
- Calculating, we obtain:

$$
\begin{gathered}
\nabla \mathbf{F}(x, y, z)=\langle 2 x, 2 y,-4 z\rangle \\
\mathbf{n}=\nabla \mathbf{F}(1,1,1)=\langle 2,2,-4\rangle
\end{gathered}
$$

## Problem 4 - Fall 2008

Consider the surface $x^{2}+y^{2}-2 z^{2}=0$ and the point $P(1,1,1)$ which lies on the surface.
(i) Find the equation of the tangent plane to the surface at $P$.
(ii) Find the equation of the normal line to the surface at $P$.

## Solution:

- Recall that the gradient of $\mathbf{F}(x, y, z)=x^{2}+y^{2}-2 z^{2}$ is normal $\mathbf{n}$ to the surface.
- Calculating, we obtain:

$$
\begin{gathered}
\nabla \mathbf{F}(x, y, z)=\langle 2 x, 2 y,-4 z\rangle \\
\mathbf{n}=\nabla \mathbf{F}(1,1,1)=\langle 2,2,-4\rangle
\end{gathered}
$$

- The equation of the tangent plane is:


## Problem 4 - Fall 2008

Consider the surface $x^{2}+y^{2}-2 z^{2}=0$ and the point $P(1,1,1)$ which lies on the surface.
(i) Find the equation of the tangent plane to the surface at $P$.
(ii) Find the equation of the normal line to the surface at $P$.

## Solution:

- Recall that the gradient of $\mathbf{F}(x, y, z)=x^{2}+y^{2}-2 z^{2}$ is normal $\mathbf{n}$ to the surface.
- Calculating, we obtain:

$$
\begin{gathered}
\nabla \mathbf{F}(x, y, z)=\langle 2 x, 2 y,-4 z\rangle \\
\mathbf{n}=\nabla \mathbf{F}(1,1,1)=\langle 2,2,-4\rangle
\end{gathered}
$$

- The equation of the tangent plane is:

$$
\langle 2,2,-4\rangle \cdot\langle x-1, y-1, z-1\rangle=2(x-1)+2(y-1)-4(z-1)=0 .
$$

## Problem 4 - Fall 2008

Consider the surface $x^{2}+y^{2}-2 z^{2}=0$ and the point $P(1,1,1)$ which lies on the surface.
(i) Find the equation of the tangent plane to the surface at $P$.
(ii) Find the equation of the normal line to the surface at $P$.

## Solution:

- Recall that the gradient of $\mathbf{F}(x, y, z)=x^{2}+y^{2}-2 z^{2}$ is normal $\mathbf{n}$ to the surface.
- Calculating, we obtain:

$$
\begin{gathered}
\nabla \mathbf{F}(x, y, z)=\langle 2 x, 2 y,-4 z\rangle \\
\mathbf{n}=\nabla \mathbf{F}(1,1,1)=\langle 2,2,-4\rangle .
\end{gathered}
$$

- The equation of the tangent plane is:

$$
\langle 2,2,-4\rangle \cdot\langle x-1, y-1, z-1\rangle=2(x-1)+2(y-1)-4(z-1)=0 .
$$

- The vector equation of the normal line is:


## Problem 4 - Fall 2008

Consider the surface $x^{2}+y^{2}-2 z^{2}=0$ and the point $P(1,1,1)$ which lies on the surface.
(i) Find the equation of the tangent plane to the surface at $P$.
(ii) Find the equation of the normal line to the surface at $P$.

## Solution:

- Recall that the gradient of $\mathbf{F}(x, y, z)=x^{2}+y^{2}-2 z^{2}$ is normal $\mathbf{n}$ to the surface.
- Calculating, we obtain:

$$
\begin{gathered}
\nabla \mathbf{F}(x, y, z)=\langle 2 x, 2 y,-4 z\rangle \\
\mathbf{n}=\nabla \mathbf{F}(1,1,1)=\langle 2,2,-4\rangle
\end{gathered}
$$

- The equation of the tangent plane is:

$$
\langle 2,2,-4\rangle \cdot\langle x-1, y-1, z-1\rangle=2(x-1)+2(y-1)-4(z-1)=0 .
$$

- The vector equation of the normal line is:

$$
\mathrm{L}(t)=\langle 1,1,1\rangle+t\langle 2,2,-4\rangle
$$

## Problem 4 - Fall 2008

Consider the surface $x^{2}+y^{2}-2 z^{2}=0$ and the point $P(1,1,1)$ which lies on the surface.
(i) Find the equation of the tangent plane to the surface at $P$.
(ii) Find the equation of the normal line to the surface at $P$.

## Solution:

- Recall that the gradient of $\mathbf{F}(x, y, z)=x^{2}+y^{2}-2 z^{2}$ is normal $\mathbf{n}$ to the surface.
- Calculating, we obtain:

$$
\begin{gathered}
\nabla \mathbf{F}(x, y, z)=\langle 2 x, 2 y,-4 z\rangle \\
\mathbf{n}=\nabla \mathbf{F}(1,1,1)=\langle 2,2,-4\rangle
\end{gathered}
$$

- The equation of the tangent plane is:

$$
\langle 2,2,-4\rangle \cdot\langle x-1, y-1, z-1\rangle=2(x-1)+2(y-1)-4(z-1)=0 .
$$

- The vector equation of the normal line is:

$$
\mathrm{L}(t)=\langle 1,1,1\rangle+t\langle 2,2,-4\rangle=\langle 1+2 t, 1+2 t, 1-4 t\rangle .
$$

## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+12 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+12 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle
$$

- This gives the following two equations:


## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+12 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle
$$

- This gives the following two equations:

$$
6 x^{2}+y^{2}+12 x=0
$$

$$
2 x y+2 y=y(2 x+2)=0
$$

## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+12 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

- This gives the following two equations:

$$
6 x^{2}+y^{2}+12 x=0
$$

$$
2 x y+2 y=y(2 x+2)=0 \Longrightarrow y=0 \text { or } x=-1
$$

## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+12 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle
$$

- This gives the following two equations:

$$
6 x^{2}+y^{2}+12 x=0
$$

$$
2 x y+2 y=y(2 x+2)=0 \Longrightarrow y=0 \text { or } x=-1
$$

- If $x=-1$, then the first equation gives:

$$
6+y^{2}-12=y^{2}-6=0
$$

## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+12 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle
$$

- This gives the following two equations:

$$
\begin{gathered}
6 x^{2}+y^{2}+12 x=0 \\
2 x y+2 y=y(2 x+2)=0 \Longrightarrow y=0 \text { or } x=-1
\end{gathered}
$$

- If $x=-1$, then the first equation gives:

$$
6+y^{2}-12=y^{2}-6=0 \Longrightarrow y=\sqrt{6} \text { or } y=-\sqrt{6}
$$

## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+12 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle
$$

- This gives the following two equations:

$$
\begin{gathered}
6 x^{2}+y^{2}+12 x=0 \\
2 x y+2 y=y(2 x+2)=0 \Longrightarrow y=0 \text { or } x=-1
\end{gathered}
$$

- If $x=-1$, then the first equation gives:

$$
6+y^{2}-12=y^{2}-6=0 \Longrightarrow y=\sqrt{6} \text { or } y=-\sqrt{6}
$$

- If $y=0$, then the first equation gives $x=0$ or $x=-2$.


## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+12 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle
$$

- This gives the following two equations:

$$
\begin{gathered}
6 x^{2}+y^{2}+12 x=0 \\
2 x y+2 y=y(2 x+2)=0 \Longrightarrow y=0 \text { or } x=-1
\end{gathered}
$$

- If $x=-1$, then the first equation gives:

$$
6+y^{2}-12=y^{2}-6=0 \Longrightarrow y=\sqrt{6} \text { or } y=-\sqrt{6}
$$

- If $y=0$, then the first equation gives $x=0$ or $x=-2$.
- The set of critical points is:

$$
\{(0,0),(-2,0),(-1, \sqrt{6}),(-1,-\sqrt{6})\} .
$$

Problem 5 - Fall 2008
Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

Solution: Continuation of problem 5.

- Recall that $\{(0,0),(-2,0),(-1, \sqrt{6}),(-1,-\sqrt{6})\}$ is the set of critical points.


## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

Solution: Continuation of problem 5.

- Recall that $\{(0,0),(-2,0),(-1, \sqrt{6}),(-1,-\sqrt{6})\}$ is the set of critical points.
- Since we will apply the Second Derivative Test, we first write down the second derivative matrix:

$$
\mathbf{D}=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x+12 & 2 y \\
2 y & 2 x+2
\end{array}\right|
$$

## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

Solution: Continuation of problem 5.

- Recall that $\{(0,0),(-2,0),(-1, \sqrt{6}),(-1,-\sqrt{6})\}$ is the set of critical points.
- Since we will apply the Second Derivative Test, we first write down the second derivative matrix:

$$
\mathbf{D}=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x+12 & 2 y \\
2 y & 2 x+2
\end{array}\right|
$$

- Since $D(0,0)=12 \cdot 2=24>0$ and $f_{x x}(0)=12<0$,


## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

Solution: Continuation of problem 5.

- Recall that $\{(0,0),(-2,0),(-1, \sqrt{6}),(-1,-\sqrt{6})\}$ is the set of critical points.
- Since we will apply the Second Derivative Test, we first write down the second derivative matrix:

$$
\mathbf{D}=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x+12 & 2 y \\
2 y & 2 x+2
\end{array}\right|
$$

- Since $D(0,0)=12 \cdot 2=24>0$ and $f_{x x}(0)=12<0$, then $(0,0)$ is a local minimum.


## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution: Continuation of problem 5.

- Recall that $\{(0,0),(-2,0),(-1, \sqrt{6}),(-1,-\sqrt{6})\}$ is the set of critical points.
- Since we will apply the Second Derivative Test, we first write down the second derivative matrix:

$$
\mathbf{D}=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x+12 & 2 y \\
2 y & 2 x+2
\end{array}\right|
$$

- Since $D(0,0)=12 \cdot 2=24>0$ and $f_{x x}(0)=12<0$, then $(0,0)$ is a local minimum.
- Since $D(-2,0)=24>0$ and $f_{x x}(-2,0)=-12>0$,


## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution: Continuation of problem 5.

- Recall that $\{(0,0),(-2,0),(-1, \sqrt{6}),(-1,-\sqrt{6})\}$ is the set of critical points.
- Since we will apply the Second Derivative Test, we first write down the second derivative matrix:

$$
\mathbf{D}=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x+12 & 2 y \\
2 y & 2 x+2
\end{array}\right|
$$

- Since $D(0,0)=12 \cdot 2=24>0$ and $f_{x x}(0)=12<0$, then $(0,0)$ is a local minimum.
- Since $D(-2,0)=24>0$ and $f_{x x}(-2,0)=-12>0$, then $(-2,0)$ is a local maximum.


## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution: Continuation of problem 5.

- Recall that $\{(0,0),(-2,0),(-1, \sqrt{6}),(-1,-\sqrt{6})\}$ is the set of critical points.
- Since we will apply the Second Derivative Test, we first write down the second derivative matrix:

$$
\mathbf{D}=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x+12 & 2 y \\
2 y & 2 x+2
\end{array}\right|
$$

- Since $D(0,0)=12 \cdot 2=24>0$ and $f_{x x}(0)=12<0$, then $(0,0)$ is a local minimum.
- Since $D(-2,0)=24>0$ and $f_{x x}(-2,0)=-12>0$, then $(-2,0)$ is a local maximum.
- Since $D(-1, \sqrt{6})<0$, then $(-1, \sqrt{6})$ is a saddle point.


## Problem 5-Fall 2008

Let $\quad f(x, y)=2 x^{3}+x y^{2}+6 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution: Continuation of problem 5.

- Recall that $\{(0,0),(-2,0),(-1, \sqrt{6}),(-1,-\sqrt{6})\}$ is the set of critical points.
- Since we will apply the Second Derivative Test, we first write down the second derivative matrix:

$$
\mathbf{D}=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x+12 & 2 y \\
2 y & 2 x+2
\end{array}\right|
$$

- Since $D(0,0)=12 \cdot 2=24>0$ and $f_{x x}(0)=12<0$, then $(0,0)$ is a local minimum.
- Since $D(-2,0)=24>0$ and $f_{x x}(-2,0)=-12>0$, then $(-2,0)$ is a local maximum.
- Since $D(-1, \sqrt{6})<0$, then $(-1, \sqrt{6})$ is a saddle point.
- Since $D(-1,-\sqrt{6})<0$, then $(-1,-\sqrt{6})$ is a saddle point. $\square$


## Problem 6 - Fall 2008

A flat circular plate has the shape of the region $x^{2}+y^{2} \leq 4$. The plate (including the boundary $x^{2}+y^{2}=4$ ) is heated so that the temperature at any point $(x, y)$ on the plate is given by $\mathrm{T}(x, y)=x^{2}+y^{2}-2 x$. Find the temperatures at the hottest and the coldest points on the plate, including the boundary $x^{2}+y^{2}=4$.

## Problem 6 - Fall 2008

A flat circular plate has the shape of the region $x^{2}+y^{2} \leq 4$. The plate (including the boundary $x^{2}+y^{2}=4$ ) is heated so that the temperature at any point $(x, y)$ on the plate is given by $\mathrm{T}(x, y)=x^{2}+y^{2}-2 x$. Find the temperatures at the hottest and the coldest points on the plate, including the boundary $x^{2}+y^{2}=4$.

## Solution:

- We first find the critical points.


## Problem 6 - Fall 2008

A flat circular plate has the shape of the region $x^{2}+y^{2} \leq 4$. The plate (including the boundary $x^{2}+y^{2}=4$ ) is heated so that the temperature at any point $(x, y)$ on the plate is given by $\mathrm{T}(x, y)=x^{2}+y^{2}-2 x$. Find the temperatures at the hottest and the coldest points on the plate, including the boundary $x^{2}+y^{2}=4$.

## Solution:

- We first find the critical points.

$$
\nabla \mathbf{T}=\langle 2 x-2,2 y\rangle=0 \Longrightarrow x=1 \text { and } y=0
$$

## Problem 6 - Fall 2008

A flat circular plate has the shape of the region $x^{2}+y^{2} \leq 4$. The plate (including the boundary $x^{2}+y^{2}=4$ ) is heated so that the temperature at any point $(x, y)$ on the plate is given by $\mathrm{T}(x, y)=x^{2}+y^{2}-2 x$. Find the temperatures at the hottest and the coldest points on the plate, including the boundary $x^{2}+y^{2}=4$.

## Solution:

- We first find the critical points.

$$
\nabla \mathbf{T}=\langle 2 x-2,2 y\rangle=0 \Longrightarrow x=1 \text { and } y=0
$$

- Next use Lagrange Multipliers to study max and $\min$ of $f$ on the boundary circle $g(x, y)=x^{2}+y^{2}=4$ :


## Problem 6 - Fall 2008

A flat circular plate has the shape of the region $x^{2}+y^{2} \leq 4$. The plate (including the boundary $x^{2}+y^{2}=4$ ) is heated so that the temperature at any point $(x, y)$ on the plate is given by $\mathrm{T}(x, y)=x^{2}+y^{2}-2 x$. Find the temperatures at the hottest and the coldest points on the plate, including the boundary $x^{2}+y^{2}=4$.

## Solution:

- We first find the critical points.

$$
\nabla \mathbf{T}=\langle 2 x-2,2 y\rangle=0 \Longrightarrow x=1 \text { and } y=0
$$

- Next use Lagrange Multipliers to study max and $\min$ of $f$ on the boundary circle $g(x, y)=x^{2}+y^{2}=4$ :

$$
\nabla \mathbf{T}=\langle 2 x-2,2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle .
$$

## Problem 6 - Fall 2008

A flat circular plate has the shape of the region $x^{2}+y^{2} \leq 4$. The plate (including the boundary $x^{2}+y^{2}=4$ ) is heated so that the temperature at any point $(x, y)$ on the plate is given by $\mathrm{T}(x, y)=x^{2}+y^{2}-2 x$. Find the temperatures at the hottest and the coldest points on the plate, including the boundary $x^{2}+y^{2}=4$.

## Solution:

- We first find the critical points.

$$
\nabla \mathbf{T}=\langle 2 x-2,2 y\rangle=0 \Longrightarrow x=1 \text { and } y=0
$$

- Next use Lagrange Multipliers to study max and $\min$ of $f$ on the boundary circle $g(x, y)=x^{2}+y^{2}=4$ : $\nabla \mathbf{T}=\langle 2 x-2,2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$.
- $2 y=\lambda 2 y \Longrightarrow y=0$ or $\lambda=1$.


## Problem 6 - Fall 2008

A flat circular plate has the shape of the region $x^{2}+y^{2} \leq 4$. The plate (including the boundary $x^{2}+y^{2}=4$ ) is heated so that the temperature at any point $(x, y)$ on the plate is given by $\mathrm{T}(x, y)=x^{2}+y^{2}-2 x$. Find the temperatures at the hottest and the coldest points on the plate, including the boundary $x^{2}+y^{2}=4$.

## Solution:

- We first find the critical points.

$$
\nabla \mathbf{T}=\langle 2 x-2,2 y\rangle=0 \Longrightarrow x=1 \text { and } y=0
$$

- Next use Lagrange Multipliers to study max and $\min$ of $f$ on the boundary circle $g(x, y)=x^{2}+y^{2}=4$ : $\nabla \mathbf{T}=\langle 2 x-2,2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$.
- $2 y=\lambda 2 y \Longrightarrow y=0$ or $\lambda=1$.
- $y=0 \Longrightarrow x= \pm 2$.


## Problem 6 - Fall 2008

A flat circular plate has the shape of the region $x^{2}+y^{2} \leq 4$. The plate (including the boundary $x^{2}+y^{2}=4$ ) is heated so that the temperature at any point $(x, y)$ on the plate is given by $\mathrm{T}(x, y)=x^{2}+y^{2}-2 x$. Find the temperatures at the hottest and the coldest points on the plate, including the boundary $x^{2}+y^{2}=4$.

## Solution:

- We first find the critical points.

$$
\nabla \mathbf{T}=\langle 2 x-2,2 y\rangle=0 \Longrightarrow x=1 \text { and } y=0
$$

- Next use Lagrange Multipliers to study max and $\min$ of $f$ on the boundary circle $g(x, y)=x^{2}+y^{2}=4$ : $\nabla \mathbf{T}=\langle 2 x-2,2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$.
- $2 y=\lambda 2 y \Longrightarrow y=0$ or $\lambda=1$.
- $y=0 \Longrightarrow x= \pm 2$.
- $\lambda=1 \Longrightarrow 2 x-2=2 x$, which is impossible.


## Problem 6 - Fall 2008

A flat circular plate has the shape of the region $x^{2}+y^{2} \leq 4$. The plate (including the boundary $x^{2}+y^{2}=4$ ) is heated so that the temperature at any point $(x, y)$ on the plate is given by $\mathrm{T}(x, y)=x^{2}+y^{2}-2 x$. Find the temperatures at the hottest and the coldest points on the plate, including the boundary $x^{2}+y^{2}=4$.

## Solution:

- We first find the critical points.

$$
\nabla \mathbf{T}=\langle 2 x-2,2 y\rangle=0 \Longrightarrow x=1 \text { and } y=0
$$

- Next use Lagrange Multipliers to study max and $\min$ of $f$ on the boundary circle $g(x, y)=x^{2}+y^{2}=4$ : $\nabla \mathbf{T}=\langle 2 x-2,2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$.
- $2 y=\lambda 2 y \Longrightarrow y=0$ or $\lambda=1$.
- $y=0 \Longrightarrow x= \pm 2$.
- $\lambda=1 \Longrightarrow 2 x-2=2 x$, which is impossible.
- Now check the value of T at 3 points:

$$
\mathbf{T}(1,0)=-1, \quad \mathbf{T}(2,0)=0, \quad \mathbf{T}(-2,0)=8 .
$$

## Problem 6 - Fall 2008

A flat circular plate has the shape of the region $x^{2}+y^{2} \leq 4$. The plate (including the boundary $x^{2}+y^{2}=4$ ) is heated so that the temperature at any point $(x, y)$ on the plate is given by $\mathrm{T}(x, y)=x^{2}+y^{2}-2 x$. Find the temperatures at the hottest and the coldest points on the plate, including the boundary $x^{2}+y^{2}=4$.

## Solution:

- We first find the critical points.

$$
\nabla \mathbf{T}=\langle 2 x-2,2 y\rangle=0 \Longrightarrow x=1 \text { and } y=0
$$

- Next use Lagrange Multipliers to study max and $\min$ of $f$ on the boundary circle $g(x, y)=x^{2}+y^{2}=4$ : $\nabla \mathbf{T}=\langle 2 x-2,2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$.
- $2 y=\lambda 2 y \Longrightarrow y=0$ or $\lambda=1$.
- $y=0 \Longrightarrow x= \pm 2$.
- $\lambda=1 \Longrightarrow 2 x-2=2 x$, which is impossible.
- Now check the value of T at 3 points:

$$
\mathrm{T}(1,0)=-1, \quad \mathrm{~T}(2,0)=0, \quad \mathrm{~T}(-2,0)=8 .
$$

- Maximum temperature is 8 and the minimum temperature is -1 .


## Problem 7(a) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$. Identify this quadric (i.e. quadratic surface), and graph the portion of the surface in the region $x \geq 0, y \geq 0$, and $z \geq 0$. Your graph should include tick marks along the three positive coordinate axes, and must clearly show where the surface intersects any of the three positive coordinate axes.

## Solution:

This is an ellipsoid. A problem of this type will not be on this midterm.

## Problem 7(b) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$. Calculate $z_{x}$ and $z_{y}$ at an arbitrary point ( $x, y, z$ ) on the surface (wherever possible).

## Problem 7(b) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$. Calculate $z_{x}$ and $z_{y}$ at an arbitrary point ( $x, y, z$ ) on the surface (wherever possible).

## Solution:

- Recall the following formulas of implicit differentiation of

$$
\mathbf{F}(x, y, z)=x^{2}+\frac{y^{2}}{9}+\frac{z^{2}}{4}-1=0
$$

## Problem 7(b) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$. Calculate $z_{x}$ and $z_{y}$ at an arbitrary point ( $x, y, z$ ) on the surface (wherever possible).

## Solution:

- Recall the following formulas of implicit differentiation of

$$
\begin{aligned}
& \mathbf{F}(x, y, z)=x^{2}+\frac{y^{2}}{9}+\frac{z^{2}}{4}-1=0: \\
& \frac{\partial z}{\partial x}=\frac{-\frac{\partial \mathbf{F}}{\partial x}}{\frac{\partial F}{\partial z}} \quad \frac{\partial z}{\partial y}=\frac{-\frac{\partial \mathbf{F}}{\partial y}}{\frac{\partial F}{\partial z}} .
\end{aligned}
$$

## Problem 7(b) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$. Calculate $z_{x}$ and $z_{y}$ at an arbitrary point ( $x, y, z$ ) on the surface (wherever possible).

## Solution:

- Recall the following formulas of implicit differentiation of

$$
\begin{aligned}
& \mathbf{F}(x, y, z)=x^{2}+\frac{y^{2}}{9}+\frac{z^{2}}{4}-1=0: \\
& \frac{\partial z}{\partial x}=\frac{-\frac{\partial \mathbf{F}}{\partial x}}{\frac{\partial \mathbf{F}}{\partial z}} \quad \frac{\partial z}{\partial y}=\frac{-\frac{\partial \mathbf{F}}{\partial y}}{\frac{\partial \mathbf{F}}{\partial z}} .
\end{aligned}
$$

- Plugging in the following values,

$$
\frac{\partial \mathbf{F}}{\partial x}=2 x \quad \frac{\partial \mathbf{F}}{\partial y}=\frac{2}{9} y \quad \frac{\partial \mathbf{F}}{\partial z}=\frac{1}{2} z
$$

## Problem 7(b) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$. Calculate $z_{x}$ and $z_{y}$ at an arbitrary point ( $x, y, z$ ) on the surface (wherever possible).

## Solution:

- Recall the following formulas of implicit differentiation of

$$
\begin{aligned}
& \mathbf{F}(x, y, z)=x^{2}+\frac{y^{2}}{9}+\frac{z^{2}}{4}-1=0: \\
& \frac{\partial z}{\partial x}=\frac{-\frac{\partial \mathbf{F}}{\partial x}}{\frac{\partial \mathbf{F}}{\partial z}} \quad \frac{\partial z}{\partial y}=\frac{-\frac{\partial \mathbf{F}}{\partial y}}{\frac{\partial \mathbf{F}}{\partial z}} .
\end{aligned}
$$

- Plugging in the following values,

$$
\frac{\partial \mathbf{F}}{\partial x}=2 x \quad \frac{\partial \mathbf{F}}{\partial y}=\frac{2}{9} y \quad \frac{\partial \mathbf{F}}{\partial z}=\frac{1}{2} z
$$

yields

$$
z_{x}=\frac{-2 x}{\frac{1}{2} z}
$$

## Problem 7(b) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$. Calculate $z_{x}$ and $z_{y}$ at an arbitrary point $(x, y, z)$ on the surface (wherever possible).

## Solution:

- Recall the following formulas of implicit differentiation of

$$
\begin{aligned}
& \mathbf{F}(x, y, z)=x^{2}+\frac{y^{2}}{9}+\frac{z^{2}}{4}-1=0: \\
& \frac{\partial z}{\partial x}=\frac{-\frac{\partial \mathbf{F}}{\partial x}}{\frac{\partial \mathbf{F}}{\partial z}} \quad \frac{\partial z}{\partial y}=\frac{-\frac{\partial \mathbf{F}}{\partial y}}{\frac{\partial \mathbf{F}}{\partial z}} .
\end{aligned}
$$

- Plugging in the following values,

$$
\frac{\partial \mathbf{F}}{\partial x}=2 x \quad \frac{\partial \mathbf{F}}{\partial y}=\frac{2}{9} y \quad \frac{\partial \mathbf{F}}{\partial z}=\frac{1}{2} z
$$

yields

$$
z_{x}=\frac{-2 x}{\frac{1}{2} z}=-4 \cdot \frac{x}{z}
$$

## Problem 7(b) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$. Calculate $z_{x}$ and $z_{y}$ at an arbitrary point ( $x, y, z$ ) on the surface (wherever possible).

## Solution:

- Recall the following formulas of implicit differentiation of

$$
\begin{aligned}
& \mathbf{F}(x, y, z)=x^{2}+\frac{y^{2}}{9}+\frac{z^{2}}{4}-1=0: \\
& \frac{\partial z}{\partial x}=\frac{-\frac{\partial \mathbf{F}}{\partial x}}{\frac{\partial \mathbf{F}}{\partial z}} \quad \frac{\partial z}{\partial y}=\frac{-\frac{\partial \mathbf{F}}{\partial y}}{\frac{\partial \mathbf{F}}{\partial z}} .
\end{aligned}
$$

- Plugging in the following values,

$$
\frac{\partial \mathbf{F}}{\partial x}=2 x \quad \frac{\partial \mathbf{F}}{\partial y}=\frac{2}{9} y \quad \frac{\partial \mathbf{F}}{\partial z}=\frac{1}{2} z
$$

yields

$$
z_{x}=\frac{-2 x}{\frac{1}{2} z}=-4 \cdot \frac{x}{z} \quad z_{y}=\frac{-\frac{2}{9} y}{\frac{1}{2} z}
$$

## Problem 7(b) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$. Calculate $z_{x}$ and $z_{y}$ at an arbitrary point ( $x, y, z$ ) on the surface (wherever possible).

## Solution:

- Recall the following formulas of implicit differentiation of

$$
\begin{aligned}
& \mathbf{F}(x, y, z)=x^{2}+\frac{y^{2}}{9}+\frac{z^{2}}{4}-1=0: \\
& \frac{\partial z}{\partial x}=\frac{-\frac{\partial \mathbf{F}}{\partial x}}{\frac{\partial \mathbf{F}}{\partial z}} \quad \frac{\partial z}{\partial y}=\frac{-\frac{\partial \mathbf{F}}{\partial y}}{\frac{\partial \mathbf{F}}{\partial z}} .
\end{aligned}
$$

- Plugging in the following values,

$$
\frac{\partial \mathbf{F}}{\partial x}=2 x \quad \frac{\partial \mathbf{F}}{\partial y}=\frac{2}{9} y \quad \frac{\partial \mathbf{F}}{\partial z}=\frac{1}{2} z
$$

yields

$$
z_{x}=\frac{-2 x}{\frac{1}{2} z}=-4 \cdot \frac{x}{z} \quad z_{y}=\frac{-\frac{2}{9} y}{\frac{1}{2} z}=-\frac{4}{9} \cdot \frac{y}{z}
$$

## Problem 7(b) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$. Calculate $z_{x}$ and $z_{y}$ at an arbitrary point ( $x, y, z$ ) on the surface (wherever possible).

## Solution:

- Recall the following formulas of implicit differentiation of

$$
\begin{aligned}
& \mathbf{F}(x, y, z)=x^{2}+\frac{y^{2}}{9}+\frac{z^{2}}{4}-1=0: \\
& \frac{\partial z}{\partial x}=\frac{-\frac{\partial \mathbf{F}}{\partial x}}{\frac{\partial \mathbf{F}}{\partial z}} \quad \frac{\partial z}{\partial y}=\frac{-\frac{\partial \mathbf{F}}{\partial y}}{\frac{\partial \mathbf{F}}{\partial z}} .
\end{aligned}
$$

- Plugging in the following values,

$$
\frac{\partial \mathbf{F}}{\partial x}=2 x \quad \frac{\partial \mathbf{F}}{\partial y}=\frac{2}{9} y \quad \frac{\partial \mathbf{F}}{\partial z}=\frac{1}{2} z
$$

yields

$$
z_{x}=\frac{-2 x}{\frac{1}{2} z}=-4 \cdot \frac{x}{z} \quad z_{y}=\frac{-\frac{2}{9} y}{\frac{1}{2} z}=-\frac{4}{9} \cdot \frac{y}{z}
$$

which make sense when $z \neq 0$.

## Problem 7(c) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$.
Determine the equation of the tangent plane to the surface at the point $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)$.

## Problem 7(c) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$.
Determine the equation of the tangent plane to the surface at the point $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)$.

Solution:

- For $\mathbf{F}(x, y, z)=x^{2}+\frac{y^{2}}{9}+\frac{z^{2}}{4}=1$, the simplest way of finding the normal vector $\mathbf{n}$ is to use $\mathbf{n}=\nabla \mathbf{F}\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)$ :


## Problem 7(c) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$.
Determine the equation of the tangent plane to the surface at the point $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)$.

## Solution:

- For $\mathbf{F}(x, y, z)=x^{2}+\frac{y^{2}}{9}+\frac{z^{2}}{4}=1$, the simplest way of finding the normal vector $\mathbf{n}$ is to use $\mathbf{n}=\nabla \mathbf{F}\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)$ :

$$
\nabla \mathbf{F}(x, y, z)=\left\langle 2 x, \frac{2}{9} y, \frac{1}{2} z\right\rangle
$$

## Problem 7(c) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$.
Determine the equation of the tangent plane to the surface at the point $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)$.

## Solution:

- For $\mathbf{F}(x, y, z)=x^{2}+\frac{y^{2}}{9}+\frac{z^{2}}{4}=1$, the simplest way of finding the normal vector $\mathbf{n}$ is to use $\mathbf{n}=\nabla \mathbf{F}\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)$ :

$$
\begin{gathered}
\nabla \mathbf{F}(x, y, z)=\left\langle 2 x, \frac{2}{9} y, \frac{1}{2} z\right\rangle \\
\mathbf{n}=\nabla \mathbf{F}\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)=\left\langle\frac{2}{\sqrt{2}}, \frac{6}{18}, \frac{1}{2}\right\rangle=\left\langle\sqrt{2}, \frac{1}{3}, \frac{1}{2}\right\rangle .
\end{gathered}
$$

## Problem 7(c) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$.
Determine the equation of the tangent plane to the surface at the point $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)$.

## Solution:

- For $\mathbf{F}(x, y, z)=x^{2}+\frac{y^{2}}{9}+\frac{z^{2}}{4}=1$, the simplest way of finding the normal vector $\mathbf{n}$ is to use $\mathbf{n}=\nabla \mathbf{F}\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)$ :

$$
\begin{gathered}
\nabla \mathbf{F}(x, y, z)=\left\langle 2 x, \frac{2}{9} y, \frac{1}{2} z\right\rangle \\
\mathbf{n}=\nabla \mathbf{F}\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)=\left\langle\frac{2}{\sqrt{2}}, \frac{6}{18}, \frac{1}{2}\right\rangle=\left\langle\sqrt{2}, \frac{1}{3}, \frac{1}{2}\right\rangle .
\end{gathered}
$$

- The equation of the tangent plane is:

$$
0=\nabla \mathbf{F}\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right) \cdot\left\langle x-\frac{1}{\sqrt{2}}, y-\frac{3}{2}, z-1\right\rangle
$$

## Problem 7(c) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$.
Determine the equation of the tangent plane to the surface at the point $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)$.

Solution:

- For $\mathbf{F}(x, y, z)=x^{2}+\frac{y^{2}}{9}+\frac{z^{2}}{4}=1$, the simplest way of finding the normal vector $\mathbf{n}$ is to use $\mathbf{n}=\nabla \mathbf{F}\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)$ :

$$
\begin{gathered}
\nabla \mathbf{F}(x, y, z)=\left\langle 2 x, \frac{2}{9} y, \frac{1}{2} z\right\rangle \\
\mathbf{n}=\nabla \mathbf{F}\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)=\left\langle\frac{2}{\sqrt{2}}, \frac{6}{18}, \frac{1}{2}\right\rangle=\left\langle\sqrt{2}, \frac{1}{3}, \frac{1}{2}\right\rangle .
\end{gathered}
$$

- The equation of the tangent plane is:

$$
\begin{aligned}
0 & =\nabla \mathbf{F}\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right) \cdot\left\langle x-\frac{1}{\sqrt{2}}, y-\frac{3}{2}, z-1\right\rangle \\
& =\left\langle\sqrt{2}, \frac{1}{3}, \frac{1}{2}\right\rangle \cdot\left\langle x-\frac{1}{\sqrt{2}}, y-\frac{3}{2}, z-1\right\rangle
\end{aligned}
$$

## Problem 7(c) - Spring 2008

Consider the equation $x^{2}+y^{2} / 9+z^{2} / 4=1$.
Determine the equation of the tangent plane to the surface at the point $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)$.

## Solution:

- For $\mathbf{F}(x, y, z)=x^{2}+\frac{y^{2}}{9}+\frac{z^{2}}{4}=1$, the simplest way of finding the normal vector $\mathbf{n}$ is to use $\mathbf{n}=\nabla \mathbf{F}\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)$ :

$$
\begin{gathered}
\nabla \mathbf{F}(x, y, z)=\left\langle 2 x, \frac{2}{9} y, \frac{1}{2} z\right\rangle \\
\mathbf{n}=\nabla \mathbf{F}\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right)=\left\langle\frac{2}{\sqrt{2}}, \frac{6}{18}, \frac{1}{2}\right\rangle=\left\langle\sqrt{2}, \frac{1}{3}, \frac{1}{2}\right\rangle .
\end{gathered}
$$

- The equation of the tangent plane is:

$$
\begin{aligned}
0 & =\nabla \mathbf{F}\left(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1\right) \cdot\left\langle x-\frac{1}{\sqrt{2}}, y-\frac{3}{2}, z-1\right\rangle \\
& =\left\langle\sqrt{2}, \frac{1}{3}, \frac{1}{2}\right\rangle \cdot\left\langle x-\frac{1}{\sqrt{2}}, y-\frac{3}{2}, z-1\right\rangle \\
= & \sqrt{2}\left(x-\frac{1}{\sqrt{2}}\right)+\frac{1}{3}\left(y-\frac{3}{2}\right)+\frac{1}{2}(z-1)=0 .
\end{aligned}
$$

## Problem 8(a) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Find the linearization $\mathrm{L}(x, y)$ of $f$ at the point $(0,5)$ and use it to approximate the value of $f$ at the point (.1,4.9). (An unsupported numerical approximation to $f(.1,4.9)$ will not receive credit.)

## Problem 8(a) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Find the linearization $\mathrm{L}(x, y)$ of $f$ at the point $(0,5)$ and use it to approximate the value of $f$ at the point (.1,4.9). (An unsupported numerical approximation to $f(.1,4.9)$ will not receive credit.)

## Solution:

- Calculating partial derivatives at $(0,5)$, we obtain:

$$
\begin{gathered}
f_{x}(x, y)=2 x y+y^{2} e^{x y} \quad f_{y}(x, y)=x^{2}+e^{x y}+x y e^{x} y \\
f_{x}(0,5)=25 \\
f_{y}(0,5)=1 .
\end{gathered}
$$

## Problem 8(a) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Find the linearization $\mathrm{L}(x, y)$ of $f$ at the point $(0,5)$ and use it to approximate the value of $f$ at the point (.1,4.9). (An unsupported numerical approximation to $f(.1,4.9)$ will not receive credit.)

## Solution:

- Calculating partial derivatives at $(0,5)$, we obtain:

$$
\begin{gathered}
f_{x}(x, y)=2 x y+y^{2} e^{x y} \quad f_{y}(x, y)=x^{2}+e^{x y}+x y e^{x} y \\
f_{x}(0,5)=25 \quad f_{y}(0,5)=1 .
\end{gathered}
$$

- Let $\mathbf{L}(x, y)$ be the linear approximation at $(0,5)$.


## Problem 8(a) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Find the linearization $\mathrm{L}(x, y)$ of $f$ at the point $(0,5)$ and use it to approximate the value of $f$ at the point (.1,4.9). (An unsupported numerical approximation to $f(.1,4.9)$ will not receive credit.)

## Solution:

- Calculating partial derivatives at $(0,5)$, we obtain:

$$
\begin{gathered}
f_{x}(x, y)=2 x y+y^{2} e^{x y} \quad f_{y}(x, y)=x^{2}+e^{x y}+x y e^{x} y \\
f_{x}(0,5)=25 \quad f_{y}(0,5)=1 .
\end{gathered}
$$

- Let $\mathbf{L}(x, y)$ be the linear approximation at $(0,5)$.

$$
\mathrm{L}(x, y)=f(0,5)+f_{x}(0,5)(x-0)+f_{y}(0,5)(y-5)
$$

## Problem 8(a) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Find the linearization $\mathrm{L}(x, y)$ of $f$ at the point $(0,5)$ and use it to approximate the value of $f$ at the point (.1,4.9). (An unsupported numerical approximation to $f(.1,4.9)$ will not receive credit.)

## Solution:

- Calculating partial derivatives at $(0,5)$, we obtain:

$$
\begin{gathered}
f_{x}(x, y)=2 x y+y^{2} e^{x y} \quad f_{y}(x, y)=x^{2}+e^{x y}+x y e^{x} y \\
f_{x}(0,5)=25 \\
f_{y}(0,5)=1 .
\end{gathered}
$$

- Let $\mathrm{L}(x, y)$ be the linear approximation at $(0,5)$.

$$
\begin{gathered}
\mathbf{L}(x, y)=f(0,5)+f_{x}(0,5)(x-0)+f_{y}(0,5)(y-5) \\
\mathbf{L}(x, y)=5+25 x+(y-5) .
\end{gathered}
$$

## Problem 8(a) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Find the linearization $\mathrm{L}(x, y)$ of $f$ at the point $(0,5)$ and use it to approximate the value of $f$ at the point (.1,4.9). (An unsupported numerical approximation to $f(.1,4.9)$ will not receive credit.)

## Solution:

- Calculating partial derivatives at $(0,5)$, we obtain:

$$
\begin{gathered}
f_{x}(x, y)=2 x y+y^{2} e^{x y} \quad f_{y}(x, y)=x^{2}+e^{x y}+x y e^{x} y \\
f_{x}(0,5)=25 \\
f_{y}(0,5)=1 .
\end{gathered}
$$

- Let $\mathrm{L}(x, y)$ be the linear approximation at $(0,5)$.

$$
\begin{gathered}
\mathbf{L}(x, y)=f(0,5)+f_{x}(0,5)(x-0)+f_{y}(0,5)(y-5) \\
\mathbf{L}(x, y)=5+25 x+(y-5) .
\end{gathered}
$$

- Calculating at (.1, 4.9):

$$
\mathrm{L}(.1,4.9)
$$

## Problem 8(a) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Find the linearization $\mathrm{L}(x, y)$ of $f$ at the point $(0,5)$ and use it to approximate the value of $f$ at the point (.1,4.9). (An unsupported numerical approximation to $f(.1,4.9)$ will not receive credit.)

## Solution:

- Calculating partial derivatives at $(0,5)$, we obtain:

$$
\begin{gathered}
f_{x}(x, y)=2 x y+y^{2} e^{x y} \quad f_{y}(x, y)=x^{2}+e^{x y}+x y e^{x} y \\
f_{x}(0,5)=25 \quad f_{y}(0,5)=1 .
\end{gathered}
$$

- Let $\mathrm{L}(x, y)$ be the linear approximation at $(0,5)$.

$$
\begin{gathered}
\mathbf{L}(x, y)=f(0,5)+f_{x}(0,5)(x-0)+f_{y}(0,5)(y-5) \\
\mathbf{L}(x, y)=5+25 x+(y-5) .
\end{gathered}
$$

- Calculating at (.1, 4.9):

$$
\mathrm{L}(.1,4.9)=5+25(0.1)+(4.9-5)
$$

## Problem 8(a) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Find the linearization $\mathrm{L}(x, y)$ of $f$ at the point $(0,5)$ and use it to approximate the value of $f$ at the point (.1,4.9). (An unsupported numerical approximation to $f(.1,4.9)$ will not receive credit.)

## Solution:

- Calculating partial derivatives at $(0,5)$, we obtain:

$$
\begin{gathered}
f_{x}(x, y)=2 x y+y^{2} e^{x y} \quad f_{y}(x, y)=x^{2}+e^{x y}+x y e^{x} y \\
f_{x}(0,5)=25 \quad f_{y}(0,5)=1 .
\end{gathered}
$$

- Let $\mathrm{L}(x, y)$ be the linear approximation at $(0,5)$.

$$
\begin{gathered}
\mathbf{L}(x, y)=f(0,5)+f_{x}(0,5)(x-0)+f_{y}(0,5)(y-5) \\
\mathbf{L}(x, y)=5+25 x+(y-5) .
\end{gathered}
$$

- Calculating at (.1, 4.9):

$$
\mathrm{L}(.1,4.9)=5+25(0.1)+(4.9-5)=7.4
$$

## Problem 8(b) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Suppose that $x(r, \theta)=r \cos \theta$ and $y(r, \theta)=r \sin \theta$. Calculate $f_{\theta}$ at $r=5$ and $\theta=\frac{\pi}{2}$.

## Problem 8(b) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Suppose that $x(r, \theta)=r \cos \theta$ and $y(r, \theta)=r \sin \theta$. Calculate $f_{\theta}$ at $r=5$ and $\theta=\frac{\pi}{2}$.

## Solution:

- The Chain Rule gives

$$
\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}
$$

## Problem 8(b) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Suppose that $x(r, \theta)=r \cos \theta$ and $y(r, \theta)=r \sin \theta$. Calculate $f_{\theta}$ at $r=5$ and $\theta=\frac{\pi}{2}$.

## Solution:

- The Chain Rule gives

$$
\begin{gathered}
\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} \\
=\left(2 x y+y^{2} e^{x y}\right)(-r \sin \theta)+\left(x^{2}+e^{x y}+x y e^{x y}\right)(r \cos \theta) .
\end{gathered}
$$

## Problem 8(b) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Suppose that $x(r, \theta)=r \cos \theta$ and $y(r, \theta)=r \sin \theta$. Calculate $f_{\theta}$ at $r=5$ and $\theta=\frac{\pi}{2}$.

## Solution:

- The Chain Rule gives

$$
\begin{gathered}
\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} \\
=\left(2 x y+y^{2} e^{x y}\right)(-r \sin \theta)+\left(x^{2}+e^{x y}+x y e^{x y}\right)(r \cos \theta) .
\end{gathered}
$$

- When $r=5$ and $\theta=\frac{\pi}{2}$, then $x=0$ and $y=5$.


## Problem 8(b) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Suppose that $x(r, \theta)=r \cos \theta$ and $y(r, \theta)=r \sin \theta$. Calculate $f_{\theta}$ at $r=5$ and $\theta=\frac{\pi}{2}$.

## Solution:

- The Chain Rule gives

$$
\begin{gathered}
\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} \\
=\left(2 x y+y^{2} e^{x y}\right)(-r \sin \theta)+\left(x^{2}+e^{x y}+x y e^{x y}\right)(r \cos \theta) .
\end{gathered}
$$

- When $r=5$ and $\theta=\frac{\pi}{2}$, then $x=0$ and $y=5$.
- Thus,

$$
\frac{\partial f}{\partial \theta}
$$

## Problem 8(b) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Suppose that $x(r, \theta)=r \cos \theta$ and $y(r, \theta)=r \sin \theta$. Calculate $f_{\theta}$ at $r=5$ and $\theta=\frac{\pi}{2}$.

## Solution:

- The Chain Rule gives

$$
\begin{gathered}
\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} \\
=\left(2 x y+y^{2} e^{x y}\right)(-r \sin \theta)+\left(x^{2}+e^{x y}+x y e^{x y}\right)(r \cos \theta) .
\end{gathered}
$$

- When $r=5$ and $\theta=\frac{\pi}{2}$, then $x=0$ and $y=5$.
- Thus,

$$
\frac{\partial f}{\partial \theta}=\left(0+25 e^{0}\right)(-5)+0
$$

## Problem 8(b) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Suppose that $x(r, \theta)=r \cos \theta$ and $y(r, \theta)=r \sin \theta$. Calculate $f_{\theta}$ at $r=5$ and $\theta=\frac{\pi}{2}$.

## Solution:

- The Chain Rule gives

$$
\begin{gathered}
\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} \\
=\left(2 x y+y^{2} e^{x y}\right)(-r \sin \theta)+\left(x^{2}+e^{x y}+x y e^{x y}\right)(r \cos \theta) .
\end{gathered}
$$

- When $r=5$ and $\theta=\frac{\pi}{2}$, then $x=0$ and $y=5$.
- Thus,

$$
\frac{\partial f}{\partial \theta}=\left(0+25 e^{0}\right)(-5)+0=-125
$$

## Problem 8(c) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Suppose a particle travels along a path $(x(t), y(t))$, and that $\mathbf{F}(t)=f(x(t), y(t))$ where $f(x, y)$ is the function defined above. Calculate $\mathbf{F}^{\prime}(3)$, assuming that at time $t=3$ the particle's position is $(x(3), y(3))=(0,5)$ and its velocity is $\left(x^{\prime}(3), y^{\prime}(3)\right)=(3,-2)$.

## Problem 8(c) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Suppose a particle travels along a path $(x(t), y(t))$, and that $\mathbf{F}(t)=f(x(t), y(t))$ where $f(x, y)$ is the function defined above.
Calculate $\mathbf{F}^{\prime}(3)$, assuming that at time $t=3$ the particle's position is $(x(3), y(3))=(0,5)$ and its velocity is $\left(x^{\prime}(3), y^{\prime}(3)\right)=(3,-2)$.

## Solution:

- The Chain Rule gives

$$
\frac{d \mathbf{F}}{d t}=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}
$$

## Problem 8(c) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Suppose a particle travels along a path $(x(t), y(t))$, and that $\mathbf{F}(t)=f(x(t), y(t))$ where $f(x, y)$ is the function defined above.
Calculate $\mathbf{F}^{\prime}(3)$, assuming that at time $t=3$ the particle's position is $(x(3), y(3))=(0,5)$ and its velocity is $\left(x^{\prime}(3), y^{\prime}(3)\right)=(3,-2)$.

## Solution:

- The Chain Rule gives

$$
\frac{d \mathbf{F}}{d t}=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\left(2 x y+y^{2} e^{x y}\right) \frac{d x}{d t}+\left(x^{2}+e^{x y}+x y e^{x y}\right) \frac{d y}{d t} .
$$

## Problem 8(c) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Suppose a particle travels along a path $(x(t), y(t))$, and that $\mathbf{F}(t)=f(x(t), y(t))$ where $f(x, y)$ is the function defined above.
Calculate $\mathbf{F}^{\prime}(3)$, assuming that at time $t=3$ the particle's position is $(x(3), y(3))=(0,5)$ and its velocity is $\left(x^{\prime}(3), y^{\prime}(3)\right)=(3,-2)$.

## Solution:

- The Chain Rule gives

$$
\frac{d \mathbf{F}}{d t}=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\left(2 x y+y^{2} e^{x y}\right) \frac{d x}{d t}+\left(x^{2}+e^{x y}+x y e^{x y}\right) \frac{d y}{d t} .
$$

- Plugging in values, we obtain: $F^{\prime}(3)$


## Problem 8(c) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Suppose a particle travels along a path $(x(t), y(t))$, and that $\mathbf{F}(t)=f(x(t), y(t))$ where $f(x, y)$ is the function defined above.
Calculate $\mathbf{F}^{\prime}(3)$, assuming that at time $t=3$ the particle's position is $(x(3), y(3))=(0,5)$ and its velocity is $\left(x^{\prime}(3), y^{\prime}(3)\right)=(3,-2)$.

## Solution:

- The Chain Rule gives

$$
\frac{d \mathbf{F}}{d t}=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\left(2 x y+y^{2} e^{x y}\right) \frac{d x}{d t}+\left(x^{2}+e^{x y}+x y e^{x y}\right) \frac{d y}{d t} .
$$

- Plugging in values, we obtain:

$$
\mathbf{F}^{\prime}(3)=\left(0+5^{2} e^{0}\right)(3)+\left(0+e^{0}+0\right)(-2)
$$

## Problem 8(c) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Suppose a particle travels along a path $(x(t), y(t))$, and that $\mathbf{F}(t)=f(x(t), y(t))$ where $f(x, y)$ is the function defined above.
Calculate $\mathbf{F}^{\prime}(3)$, assuming that at time $t=3$ the particle's position is $(x(3), y(3))=(0,5)$ and its velocity is $\left(x^{\prime}(3), y^{\prime}(3)\right)=(3,-2)$.

## Solution:

- The Chain Rule gives

$$
\frac{d \mathbf{F}}{d t}=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\left(2 x y+y^{2} e^{x y}\right) \frac{d x}{d t}+\left(x^{2}+e^{x y}+x y e^{x y}\right) \frac{d y}{d t} .
$$

- Plugging in values, we obtain:

$$
\mathbf{F}^{\prime}(3)=\left(0+5^{2} e^{0}\right)(3)+\left(0+e^{0}+0\right)(-2)=75+(-2)
$$

## Problem 8(c) - Spring 2008

Given the function $f(x, y)=x^{2} y+y e^{x y}$.
Suppose a particle travels along a path $(x(t), y(t))$, and that $\mathbf{F}(t)=f(x(t), y(t))$ where $f(x, y)$ is the function defined above.
Calculate $\mathbf{F}^{\prime}(3)$, assuming that at time $t=3$ the particle's position is $(x(3), y(3))=(0,5)$ and its velocity is $\left(x^{\prime}(3), y^{\prime}(3)\right)=(3,-2)$.

## Solution:

- The Chain Rule gives

$$
\frac{d \mathbf{F}}{d t}=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\left(2 x y+y^{2} e^{x y}\right) \frac{d x}{d t}+\left(x^{2}+e^{x y}+x y e^{x y}\right) \frac{d y}{d t} .
$$

- Plugging in values, we obtain:

$$
\mathbf{F}^{\prime}(3)=\left(0+5^{2} e^{0}\right)(3)+\left(0+e^{0}+0\right)(-2)=75+(-2)=73 .
$$

## Problem 9(a) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find the directional derivative of $f(x, y)$ at $P=(-2,3)$ in the direction starting from $P$ pointing towards $Q=(0,4)$.

## Problem 9(a) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find the directional derivative of $f(x, y)$ at $P=(-2,3)$ in the direction starting from $P$ pointing towards $Q=(0,4)$.

## Solution:

- First calculate partial derivatives of $f(x, y)=2\left(x^{2}+4 y\right)^{\frac{1}{2}}$ :

$$
f_{x}=\frac{2 x}{\sqrt{x^{2}+4 y}} \quad f_{y}=\frac{4}{\sqrt{x^{2}+4 y}} .
$$

## Problem 9(a) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find the directional derivative of $f(x, y)$ at $P=(-2,3)$ in the direction starting from $P$ pointing towards $Q=(0,4)$.

## Solution:

- First calculate partial derivatives of $f(x, y)=2\left(x^{2}+4 y\right)^{\frac{1}{2}}$ :

$$
f_{x}=\frac{2 x}{\sqrt{x^{2}+4 y}} \quad f_{y}=\frac{4}{\sqrt{x^{2}+4 y}} .
$$

- So,

$$
\nabla f(-2,3)=\left\langle\frac{-4}{\sqrt{16}}, \frac{4}{\sqrt{16}}\right\rangle
$$

## Problem 9(a) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find the directional derivative of $f(x, y)$ at $P=(-2,3)$ in the direction starting from $P$ pointing towards $Q=(0,4)$.

## Solution:

- First calculate partial derivatives of $f(x, y)=2\left(x^{2}+4 y\right)^{\frac{1}{2}}$ :

$$
f_{x}=\frac{2 x}{\sqrt{x^{2}+4 y}} \quad f_{y}=\frac{4}{\sqrt{x^{2}+4 y}} .
$$

- So,

$$
\nabla f(-2,3)=\left\langle\frac{-4}{\sqrt{16}}, \frac{4}{\sqrt{16}}\right\rangle=\langle-1,1\rangle
$$

## Problem 9(a) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find the directional derivative of $f(x, y)$ at $P=(-2,3)$ in the direction starting from $P$ pointing towards $Q=(0,4)$.

## Solution:

- First calculate partial derivatives of $f(x, y)=2\left(x^{2}+4 y\right)^{\frac{1}{2}}$ :

$$
f_{x}=\frac{2 x}{\sqrt{x^{2}+4 y}} \quad f_{y}=\frac{4}{\sqrt{x^{2}+4 y}}
$$

- So,

$$
\nabla f(-2,3)=\left\langle\frac{-4}{\sqrt{16}}, \frac{4}{\sqrt{16}}\right\rangle=\langle-1,1\rangle
$$

- The unit vector $\mathbf{u}$ in the direction $\overrightarrow{P Q}=\langle 2,1\rangle$ is $\mathbf{u}=\frac{1}{\sqrt{5}}\langle 2,1\rangle$.


## Problem 9(a) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find the directional derivative of $f(x, y)$ at $P=(-2,3)$ in the direction starting from $P$ pointing towards $Q=(0,4)$.

## Solution:

- First calculate partial derivatives of $f(x, y)=2\left(x^{2}+4 y\right)^{\frac{1}{2}}$ :

$$
f_{x}=\frac{2 x}{\sqrt{x^{2}+4 y}} \quad f_{y}=\frac{4}{\sqrt{x^{2}+4 y}}
$$

- So,

$$
\nabla f(-2,3)=\left\langle\frac{-4}{\sqrt{16}}, \frac{4}{\sqrt{16}}\right\rangle=\langle-1,1\rangle
$$

- The unit vector $\mathbf{u}$ in the direction $\overrightarrow{P Q}=\langle 2,1\rangle$ is $\mathbf{u}=\frac{1}{\sqrt{5}}\langle 2,1\rangle$.
- $D_{\mathbf{u}} f(-2,3)=\nabla f(-2,3) \cdot \mathbf{u}$


## Problem 9(a) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find the directional derivative of $f(x, y)$ at $P=(-2,3)$ in the direction starting from $P$ pointing towards $Q=(0,4)$.

## Solution:

- First calculate partial derivatives of $f(x, y)=2\left(x^{2}+4 y\right)^{\frac{1}{2}}$ :

$$
f_{x}=\frac{2 x}{\sqrt{x^{2}+4 y}} \quad f_{y}=\frac{4}{\sqrt{x^{2}+4 y}}
$$

- So,

$$
\nabla f(-2,3)=\left\langle\frac{-4}{\sqrt{16}}, \frac{4}{\sqrt{16}}\right\rangle=\langle-1,1\rangle
$$

- The unit vector $\mathbf{u}$ in the direction $\overrightarrow{P Q}=\langle 2,1\rangle$ is $\mathbf{u}=\frac{1}{\sqrt{5}}\langle 2,1\rangle$.
- $D_{\mathbf{u}} f(-2,3)=\nabla f(-2,3) \cdot \mathbf{u}=\langle-1,1\rangle \cdot \frac{1}{\sqrt{5}}\langle 2,1\rangle$


## Problem 9(a) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find the directional derivative of $f(x, y)$ at $P=(-2,3)$ in the direction starting from $P$ pointing towards $Q=(0,4)$.

## Solution:

- First calculate partial derivatives of $f(x, y)=2\left(x^{2}+4 y\right)^{\frac{1}{2}}$ :

$$
f_{x}=\frac{2 x}{\sqrt{x^{2}+4 y}} \quad f_{y}=\frac{4}{\sqrt{x^{2}+4 y}}
$$

- So,

$$
\nabla f(-2,3)=\left\langle\frac{-4}{\sqrt{16}}, \frac{4}{\sqrt{16}}\right\rangle=\langle-1,1\rangle
$$

- The unit vector $\mathbf{u}$ in the direction $\overrightarrow{P Q}=\langle 2,1\rangle$ is $\mathbf{u}=\frac{1}{\sqrt{5}}\langle 2,1\rangle$.
- $D_{\mathbf{u}} f(-2,3)=\nabla f(-2,3) \cdot \mathbf{u}=\langle-1,1\rangle \cdot \frac{1}{\sqrt{5}}\langle 2,1\rangle=-\frac{1}{\sqrt{5}}$.

Problem 9(b) - Spring 2008
Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find all unit vectors $\mathbf{u}$ for which the directional derivative
$D_{\mathrm{u}} f(-2,3)=0$.

## Problem 9(b) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find all unit vectors $\mathbf{u}$ for which the directional derivative
$D_{\mathrm{u}} f(-2,3)=0$.

## Solution:

- First find all the possible non-unit vectors $\mathbf{v}=\langle x, y\rangle$ which are orthogonal to $\nabla f(-2,3)=\langle-1,1\rangle$ :


## Problem 9(b) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find all unit vectors $\mathbf{u}$ for which the directional derivative
$D_{\mathrm{u}} f(-2,3)=0$.

## Solution:

- First find all the possible non-unit vectors $\mathbf{v}=\langle x, y\rangle$ which are orthogonal to $\nabla f(-2,3)=\langle-1,1\rangle$ :

$$
\langle-1,1\rangle \cdot\langle x, y\rangle=-x+y=0
$$

## Problem 9(b) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find all unit vectors $\mathbf{u}$ for which the directional derivative
$D_{\mathrm{u}} f(-2,3)=0$.

## Solution:

- First find all the possible non-unit vectors $\mathbf{v}=\langle x, y\rangle$ which are orthogonal to $\nabla f(-2,3)=\langle-1,1\rangle$ :

$$
\langle-1,1\rangle \cdot\langle x, y\rangle=-x+y=0 \Longrightarrow x=y
$$

## Problem 9(b) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find all unit vectors $\mathbf{u}$ for which the directional derivative
$D_{\mathrm{u}} f(-2,3)=0$.

## Solution:

- First find all the possible non-unit vectors $\mathbf{v}=\langle x, y\rangle$ which are orthogonal to $\nabla f(-2,3)=\langle-1,1\rangle$ :

$$
\langle-1,1\rangle \cdot\langle x, y\rangle=-x+y=0 \Longrightarrow x=y
$$

- Therefore, $\mathbf{v}=\langle x, x\rangle$ works for any $x \neq 0$.


## Problem 9(b) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find all unit vectors $\mathbf{u}$ for which the directional derivative
$D_{\mathrm{u}} f(-2,3)=0$.

## Solution:

- First find all the possible non-unit vectors $\mathbf{v}=\langle x, y\rangle$ which are orthogonal to $\nabla f(-2,3)=\langle-1,1\rangle$ :

$$
\langle-1,1\rangle \cdot\langle x, y\rangle=-x+y=0 \Longrightarrow x=y .
$$

- Therefore, $\mathbf{v}=\langle x, x\rangle$ works for any $x \neq 0$.
- The set of unit vectors $\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}$ such that $D_{\mathbf{u}} f(-2,3)=\nabla f(-2,3) \cdot \mathbf{u}=0$ consists of 2 vectors:


## Problem 9(b) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Find all unit vectors $\mathbf{u}$ for which the directional derivative
$D_{\mathrm{u}} f(-2,3)=0$.

## Solution:

- First find all the possible non-unit vectors $\mathbf{v}=\langle x, y\rangle$ which are orthogonal to $\nabla f(-2,3)=\langle-1,1\rangle$ :

$$
\langle-1,1\rangle \cdot\langle x, y\rangle=-x+y=0 \Longrightarrow x=y .
$$

- Therefore, $\mathbf{v}=\langle x, x\rangle$ works for any $x \neq 0$.
- The set of unit vectors $\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}$ such that $D_{\mathrm{u}} f(-2,3)=\nabla f(-2,3) \cdot \mathbf{u}=0$ consists of 2 vectors:

$$
\left\{\left\langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle,\left\langle-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\rangle\right\} .
$$

## Problem 9(c) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Is there a unit vector $\mathbf{u}$ for which the directional derivative $D_{\mathrm{u}} f(-2,3)=4$ ? Either find the appropriate $\mathbf{u}$ or explain why not.

## Problem 9(c) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Is there a unit vector $\mathbf{u}$ for which the directional derivative $D_{\mathrm{u}} f(-2,3)=4$ ? Either find the appropriate $\mathbf{u}$ or explain why not.

## Solution:

- First recall that:

$$
\nabla f=\left\langle\frac{2 x}{\sqrt{x^{2}+4 y}}, \frac{4}{\sqrt{x^{2}+4 y}}\right\rangle \quad \nabla f(-2,3)=\langle-1,1\rangle .
$$

## Problem 9(c) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Is there a unit vector $\mathbf{u}$ for which the directional derivative $D_{\mathrm{u}} f(-2,3)=4$ ? Either find the appropriate $\mathbf{u}$ or explain why not.

## Solution:

- First recall that:

$$
\nabla f=\left\langle\frac{2 x}{\sqrt{x^{2}+4 y}}, \frac{4}{\sqrt{x^{2}+4 y}}\right\rangle \quad \nabla f(-2,3)=\langle-1,1\rangle .
$$

- This question is equivalent to asking whether there is a unit vector $\mathbf{u}=\langle x, y\rangle$ such that

$$
D_{\mathbf{u}} f(-2,3)=\nabla f(-2,3) \cdot \mathbf{u}
$$

## Problem 9(c) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Is there a unit vector $\mathbf{u}$ for which the directional derivative $D_{\mathrm{u}} f(-2,3)=4$ ? Either find the appropriate $\mathbf{u}$ or explain why not.

## Solution:

- First recall that:

$$
\nabla f=\left\langle\frac{2 x}{\sqrt{x^{2}+4 y}}, \frac{4}{\sqrt{x^{2}+4 y}}\right\rangle \quad \nabla f(-2,3)=\langle-1,1\rangle .
$$

- This question is equivalent to asking whether there is a unit vector $\mathbf{u}=\langle x, y\rangle$ such that

$$
D_{\mathbf{u}} f(-2,3)=\nabla f(-2,3) \cdot \mathbf{u}=\langle-1,1\rangle \cdot \mathbf{u}
$$

## Problem 9(c) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Is there a unit vector $\mathbf{u}$ for which the directional derivative $D_{\mathrm{u}} f(-2,3)=4$ ? Either find the appropriate $\mathbf{u}$ or explain why not.

## Solution:

- First recall that:

$$
\nabla f=\left\langle\frac{2 x}{\sqrt{x^{2}+4 y}}, \frac{4}{\sqrt{x^{2}+4 y}}\right\rangle \quad \nabla f(-2,3)=\langle-1,1\rangle .
$$

- This question is equivalent to asking whether there is a unit vector $\mathbf{u}=\langle x, y\rangle$ such that

$$
D_{\mathbf{u}} f(-2,3)=\nabla f(-2,3) \cdot \mathbf{u}=\langle-1,1\rangle \cdot \mathbf{u}=4 .
$$

## Problem 9(c) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Is there a unit vector $\mathbf{u}$ for which the directional derivative $D_{\mathrm{u}} f(-2,3)=4$ ? Either find the appropriate $\mathbf{u}$ or explain why not.

## Solution:

- First recall that:

$$
\nabla f=\left\langle\frac{2 x}{\sqrt{x^{2}+4 y}}, \frac{4}{\sqrt{x^{2}+4 y}}\right\rangle \quad \nabla f(-2,3)=\langle-1,1\rangle .
$$

- This question is equivalent to asking whether there is a unit vector $\mathbf{u}=\langle x, y\rangle$ such that

$$
D_{\mathbf{u}} f(-2,3)=\nabla f(-2,3) \cdot \mathbf{u}=\langle-1,1\rangle \cdot \mathbf{u}=4 .
$$

- If such $\mathbf{u}$ exists, then

$$
4=|\langle-1,1\rangle \cdot \mathbf{u}|
$$

## Problem 9(c) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Is there a unit vector $\mathbf{u}$ for which the directional derivative $D_{\mathrm{u}} f(-2,3)=4$ ? Either find the appropriate $\mathbf{u}$ or explain why not.

## Solution:

- First recall that:

$$
\nabla f=\left\langle\frac{2 x}{\sqrt{x^{2}+4 y}}, \frac{4}{\sqrt{x^{2}+4 y}}\right\rangle \quad \nabla f(-2,3)=\langle-1,1\rangle .
$$

- This question is equivalent to asking whether there is a unit vector $\mathbf{u}=\langle x, y\rangle$ such that

$$
D_{\mathbf{u}} f(-2,3)=\nabla f(-2,3) \cdot \mathbf{u}=\langle-1,1\rangle \cdot \mathbf{u}=4 .
$$

- If such $\mathbf{u}$ exists, then

$$
4=|\langle-1,1\rangle \cdot \mathbf{u}|=|\langle-1,1\rangle| \cdot|\mathbf{u}||\cos \theta|
$$

## Problem 9(c) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Is there a unit vector $\mathbf{u}$ for which the directional derivative $D_{\mathrm{u}} f(-2,3)=4$ ? Either find the appropriate $\mathbf{u}$ or explain why not.

## Solution:

- First recall that:

$$
\nabla f=\left\langle\frac{2 x}{\sqrt{x^{2}+4 y}}, \frac{4}{\sqrt{x^{2}+4 y}}\right\rangle \quad \nabla f(-2,3)=\langle-1,1\rangle .
$$

- This question is equivalent to asking whether there is a unit vector $\mathbf{u}=\langle x, y\rangle$ such that

$$
D_{\mathbf{u}} f(-2,3)=\nabla f(-2,3) \cdot \mathbf{u}=\langle-1,1\rangle \cdot \mathbf{u}=4 .
$$

- If such $\mathbf{u}$ exists, then

$$
4=|\langle-1,1\rangle \cdot \mathbf{u}|=|\langle-1,1\rangle| \cdot|\mathbf{u}||\cos \theta|=\sqrt{2}|\cos \theta|
$$

## Problem 9(c) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Is there a unit vector $\mathbf{u}$ for which the directional derivative $D_{\mathrm{u}} f(-2,3)=4$ ? Either find the appropriate $\mathbf{u}$ or explain why not.

## Solution:

- First recall that:

$$
\nabla f=\left\langle\frac{2 x}{\sqrt{x^{2}+4 y}}, \frac{4}{\sqrt{x^{2}+4 y}}\right\rangle \quad \nabla f(-2,3)=\langle-1,1\rangle .
$$

- This question is equivalent to asking whether there is a unit vector $\mathbf{u}=\langle x, y\rangle$ such that

$$
D_{\mathbf{u}} f(-2,3)=\nabla f(-2,3) \cdot \mathbf{u}=\langle-1,1\rangle \cdot \mathbf{u}=4 .
$$

- If such $\mathbf{u}$ exists, then

$$
4=|\langle-1,1\rangle \cdot \mathbf{u}|=|\langle-1,1\rangle| \cdot|\mathbf{u}||\cos \theta|=\sqrt{2}|\cos \theta| \leq \sqrt{2}
$$

## Problem 9(c) - Spring 2008

Consider the function $f(x, y)=2 \sqrt{x^{2}+4 y}$.
Is there a unit vector $\mathbf{u}$ for which the directional derivative $D_{\mathrm{u}} f(-2,3)=4$ ? Either find the appropriate $\mathbf{u}$ or explain why not.

## Solution:

- First recall that:

$$
\nabla f=\left\langle\frac{2 x}{\sqrt{x^{2}+4 y}}, \frac{4}{\sqrt{x^{2}+4 y}}\right\rangle \quad \nabla f(-2,3)=\langle-1,1\rangle .
$$

- This question is equivalent to asking whether there is a unit vector $\mathbf{u}=\langle x, y\rangle$ such that

$$
D_{\mathbf{u}} f(-2,3)=\nabla f(-2,3) \cdot \mathbf{u}=\langle-1,1\rangle \cdot \mathbf{u}=4 .
$$

- If such $\mathbf{u}$ exists, then

$$
4=|\langle-1,1\rangle \cdot \mathbf{u}|=|\langle-1,1\rangle| \cdot|\mathbf{u}||\cos \theta|=\sqrt{2}|\cos \theta| \leq \sqrt{2}
$$

- Therefore, no such unit vector u exists.

Problem 10(a) - Spring 2008
Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Find all critical points of $f(x, y)$.

Problem 10(a) - Spring 2008
Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Find all critical points of $f(x, y)$.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $\langle 0,0\rangle$ :

Problem 10(a) - Spring 2008
Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Find all critical points of $f(x, y)$.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 2 x^{2}-y, y^{2}-x\right\rangle=\langle 0,0\rangle
$$

## Problem 10(a) - Spring 2008

Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Find all critical points of $f(x, y)$.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 2 x^{2}-y, y^{2}-x\right\rangle=\langle 0,0\rangle
$$

- The first coordinate equation $2 x^{2}-y=0$ implies $y=2 x^{2}$.


## Problem 10(a) - Spring 2008

Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Find all critical points of $f(x, y)$.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 2 x^{2}-y, y^{2}-x\right\rangle=\langle 0,0\rangle
$$

- The first coordinate equation $2 x^{2}-y=0$ implies $y=2 x^{2}$.
- Plugging $y=2 x^{2}$ into the second coordinate equation gives

$$
4 x^{4}-x=x\left(4 x^{3}-1\right)=0
$$

## Problem 10(a) - Spring 2008

Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Find all critical points of $f(x, y)$.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 2 x^{2}-y, y^{2}-x\right\rangle=\langle 0,0\rangle
$$

- The first coordinate equation $2 x^{2}-y=0$ implies $y=2 x^{2}$.
- Plugging $y=2 x^{2}$ into the second coordinate equation gives $4 x^{4}-x=x\left(4 x^{3}-1\right)=0 \Longrightarrow x=0$ or $x=4^{-\frac{1}{3}}$.


## Problem 10(a) - Spring 2008

Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Find all critical points of $f(x, y)$.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 2 x^{2}-y, y^{2}-x\right\rangle=\langle 0,0\rangle
$$

- The first coordinate equation $2 x^{2}-y=0$ implies $y=2 x^{2}$.
- Plugging $y=2 x^{2}$ into the second coordinate equation gives $4 x^{4}-x=x\left(4 x^{3}-1\right)=0 \Longrightarrow x=0$ or $x=4^{-\frac{1}{3}}$.
- Hence, $(x=0$ and $y=0)$ or $\left(x=4^{-\frac{1}{3}}\right.$ and $\left.y=2 \cdot 4^{-\frac{2}{3}}\right)$.


## Problem 10(a) - Spring 2008

Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Find all critical points of $f(x, y)$.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 2 x^{2}-y, y^{2}-x\right\rangle=\langle 0,0\rangle
$$

- The first coordinate equation $2 x^{2}-y=0$ implies $y=2 x^{2}$.
- Plugging $y=2 x^{2}$ into the second coordinate equation gives $4 x^{4}-x=x\left(4 x^{3}-1\right)=0 \Longrightarrow x=0$ or $x=4^{-\frac{1}{3}}$.
- Hence, $(x=0$ and $y=0)$ or $\left(x=4^{-\frac{1}{3}}\right.$ and $\left.y=2 \cdot 4^{-\frac{2}{3}}\right)$.
- This gives a set of two critical points:

$$
\left\{(0,0),\left(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}\right)\right\}
$$

Problem 10(b) - Spring 2008
Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Classify each critical point as a relative maximum, relative (local) minimum or saddle; you do not need to calculate the function at these points, but your answer must be justified.

Problem 10(b) - Spring 2008
Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Classify each critical point as a relative maximum, relative (local) minimum or saddle; you do not need to calculate the function at these points, but your answer must be justified.

## Solution:

- By part (a) $\nabla f=\left\langle 2 x^{2}-y, y^{2}-x\right\rangle$ and the set of critical points is $\left\{(0,0),\left(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}\right)\right\}$.

Problem 10(b) - Spring 2008
Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Classify each critical point as a relative maximum, relative (local) minimum or saddle; you do not need to calculate the function at these points, but your answer must be justified.

## Solution:

- By part (a) $\nabla f=\left\langle 2 x^{2}-y, y^{2}-x\right\rangle$ and the set of critical points is $\left\{(0,0),\left(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}\right)\right\}$.
- Now write down the Hessian:

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|
$$

Problem 10(b) - Spring 2008
Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Classify each critical point as a relative maximum, relative (local) minimum or saddle; you do not need to calculate the function at these points, but your answer must be justified.

## Solution:

- By part (a) $\nabla f=\left\langle 2 x^{2}-y, y^{2}-x\right\rangle$ and the set of critical points is $\left\{(0,0),\left(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}\right)\right\}$.
- Now write down the Hessian:

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
4 x & -1 \\
-1 & 2 y
\end{array}\right|
$$

Problem 10(b) - Spring 2008
Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Classify each critical point as a relative maximum, relative (local) minimum or saddle; you do not need to calculate the function at these points, but your answer must be justified.

## Solution:

- By part (a) $\nabla f=\left\langle 2 x^{2}-y, y^{2}-x\right\rangle$ and the set of critical points is $\left\{(0,0),\left(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}\right)\right\}$.
- Now write down the Hessian:

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
4 x & -1 \\
-1 & 2 y
\end{array}\right|=8 x y-1
$$

Problem 10(b) - Spring 2008
Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Classify each critical point as a relative maximum, relative (local) minimum or saddle; you do not need to calculate the function at these points, but your answer must be justified.

## Solution:

- By part (a) $\nabla f=\left\langle 2 x^{2}-y, y^{2}-x\right\rangle$ and the set of critical points is $\left\{(0,0),\left(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}\right)\right\}$.
- Now write down the Hessian:

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
4 x & -1 \\
-1 & 2 y
\end{array}\right|=8 x y-1
$$

- Next apply the Second Derivative Test.

Problem 10(b) - Spring 2008
Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Classify each critical point as a relative maximum, relative (local) minimum or saddle; you do not need to calculate the function at these points, but your answer must be justified.

## Solution:

- By part (a) $\nabla f=\left\langle 2 x^{2}-y, y^{2}-x\right\rangle$ and the set of critical points is $\left\{(0,0),\left(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}\right)\right\}$.
- Now write down the Hessian:

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
4 x & -1 \\
-1 & 2 y
\end{array}\right|=8 x y-1
$$

- Next apply the Second Derivative Test.
- Since $D(0,0)=-1<0$, then $(0,0)$ is a saddle point.

Problem 10(b) - Spring 2008
Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Classify each critical point as a relative maximum, relative (local) minimum or saddle; you do not need to calculate the function at these points, but your answer must be justified.

## Solution:

- By part (a) $\nabla f=\left\langle 2 x^{2}-y, y^{2}-x\right\rangle$ and the set of critical points is $\left\{(0,0),\left(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}\right)\right\}$.
- Now write down the Hessian:

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
4 x & -1 \\
-1 & 2 y
\end{array}\right|=8 x y-1
$$

- Next apply the Second Derivative Test.
- Since $D(0,0)=-1<0$, then $(0,0)$ is a saddle point.
- Since $D\left(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}\right)=4-1>0$ and
$f_{x x}\left(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}\right)=4 \cdot 4^{-\frac{1}{3}}=4^{\frac{2}{3}}>0$,

Problem 10(b) - Spring 2008
Let $f(x, y)=\frac{2}{3} x^{3}+\frac{1}{3} y^{3}-x y$.
Classify each critical point as a relative maximum, relative (local) minimum or saddle; you do not need to calculate the function at these points, but your answer must be justified.

## Solution:

- By part (a) $\nabla f=\left\langle 2 x^{2}-y, y^{2}-x\right\rangle$ and the set of critical points is $\left\{(0,0),\left(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}\right)\right\}$.
- Now write down the Hessian:

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
4 x & -1 \\
-1 & 2 y
\end{array}\right|=8 x y-1
$$

- Next apply the Second Derivative Test.
- Since $D(0,0)=-1<0$, then $(0,0)$ is a saddle point.
- Since $D\left(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}\right)=4-1>0$ and $f_{x x}\left(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}\right)=4 \cdot 4^{-\frac{1}{3}}=4^{\frac{2}{3}}>0$, then $\left(4^{-\frac{1}{3}}, 2 \cdot 4^{-\frac{2}{3}}\right)$ is a local minimum.


## Problem 11-Spring 2008

Use the method of Lagrange multipliers to determine all points $(x, y)$ where the function $f(x, y)=2 x^{2}+4 y^{2}+16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{1}{4} x^{2}+y^{2}=4$.

## Problem 11 - Spring 2008

Use the method of Lagrange multipliers to determine all points $(x, y)$ where the function $f(x, y)=2 x^{2}+4 y^{2}+16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{1}{4} x^{2}+y^{2}=4$.

## Solution:

- Set $g(x, y)=\frac{1}{4} x^{2}+y^{2}$.


## Problem 11 - Spring 2008

Use the method of Lagrange multipliers to determine all points $(x, y)$ where the function $f(x, y)=2 x^{2}+4 y^{2}+16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{1}{4} x^{2}+y^{2}=4$.

## Solution:

- Set $g(x, y)=\frac{1}{4} x^{2}+y^{2}$.
- Set $\nabla f=\langle 4 x, 8 y\rangle=\lambda \nabla g=\lambda\left\langle\frac{1}{2} x, 2 y\right\rangle$ and solve:

$$
8 y=2 \lambda y
$$

## Problem 11 - Spring 2008

Use the method of Lagrange multipliers to determine all points $(x, y)$ where the function $f(x, y)=2 x^{2}+4 y^{2}+16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{1}{4} x^{2}+y^{2}=4$.

## Solution:

- Set $g(x, y)=\frac{1}{4} x^{2}+y^{2}$.
- Set $\nabla f=\langle 4 x, 8 y\rangle=\lambda \nabla g=\lambda\left\langle\frac{1}{2} x, 2 y\right\rangle$ and solve:

$$
8 y=2 \lambda y \Longrightarrow \lambda=4 \text { or } y=0 .
$$

## Problem 11 - Spring 2008

Use the method of Lagrange multipliers to determine all points $(x, y)$ where the function $f(x, y)=2 x^{2}+4 y^{2}+16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{1}{4} x^{2}+y^{2}=4$.

## Solution:

- Set $g(x, y)=\frac{1}{4} x^{2}+y^{2}$.
- Set $\nabla f=\langle 4 x, 8 y\rangle=\lambda \nabla g=\lambda\left\langle\frac{1}{2} x, 2 y\right\rangle$ and solve:

$$
\begin{aligned}
& 8 y=2 \lambda y \Longrightarrow \lambda=4 \text { or } y=0 . \\
& 4 x=\frac{1}{2} \lambda x
\end{aligned}
$$

## Problem 11 - Spring 2008

Use the method of Lagrange multipliers to determine all points $(x, y)$ where the function $f(x, y)=2 x^{2}+4 y^{2}+16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{1}{4} x^{2}+y^{2}=4$.

## Solution:

- Set $g(x, y)=\frac{1}{4} x^{2}+y^{2}$.
- Set $\nabla f=\langle 4 x, 8 y\rangle=\lambda \nabla g=\lambda\left\langle\frac{1}{2} x, 2 y\right\rangle$ and solve:

$$
\begin{aligned}
& 8 y=2 \lambda y \Longrightarrow \lambda=4 \text { or } y=0 . \\
& 4 x=\frac{1}{2} \lambda x \Longrightarrow \lambda=8 \text { or } x=0 .
\end{aligned}
$$

## Problem 11 - Spring 2008

Use the method of Lagrange multipliers to determine all points $(x, y)$ where the function $f(x, y)=2 x^{2}+4 y^{2}+16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{1}{4} x^{2}+y^{2}=4$.

## Solution:

- Set $g(x, y)=\frac{1}{4} x^{2}+y^{2}$.
- Set $\nabla f=\langle 4 x, 8 y\rangle=\lambda \nabla g=\lambda\left\langle\frac{1}{2} x, 2 y\right\rangle$ and solve:

$$
\begin{aligned}
& 8 y=2 \lambda y \Longrightarrow \lambda=4 \text { or } y=0 . \\
& 4 x=\frac{1}{2} \lambda x \Longrightarrow \lambda=8 \text { or } x=0 .
\end{aligned}
$$

- Since $\lambda$ cannot simultaneously be 4 and 8 , then $x$ or $y$ is zero.


## Problem 11 - Spring 2008

Use the method of Lagrange multipliers to determine all points $(x, y)$ where the function $f(x, y)=2 x^{2}+4 y^{2}+16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{1}{4} x^{2}+y^{2}=4$.

## Solution:

- Set $g(x, y)=\frac{1}{4} x^{2}+y^{2}$.
- Set $\nabla f=\langle 4 x, 8 y\rangle=\lambda \nabla g=\lambda\left\langle\frac{1}{2} x, 2 y\right\rangle$ and solve:

$$
\begin{aligned}
& 8 y=2 \lambda y \Longrightarrow \lambda=4 \text { or } y=0 . \\
& 4 x=\frac{1}{2} \lambda x \Longrightarrow \lambda=8 \text { or } x=0 .
\end{aligned}
$$

- Since $\lambda$ cannot simultaneously be 4 and 8 , then $x$ or $y$ is zero.
- From the constraint $\frac{1}{4} x^{2}+y^{2}=4$,


## Problem 11 - Spring 2008

Use the method of Lagrange multipliers to determine all points $(x, y)$ where the function $f(x, y)=2 x^{2}+4 y^{2}+16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{1}{4} x^{2}+y^{2}=4$.

## Solution:

- Set $g(x, y)=\frac{1}{4} x^{2}+y^{2}$.
- Set $\nabla f=\langle 4 x, 8 y\rangle=\lambda \nabla g=\lambda\left\langle\frac{1}{2} x, 2 y\right\rangle$ and solve:

$$
\begin{aligned}
& 8 y=2 \lambda y \Longrightarrow \lambda=4 \text { or } y=0 \\
& 4 x=\frac{1}{2} \lambda x \Longrightarrow \lambda=8 \text { or } x=0 .
\end{aligned}
$$

- Since $\lambda$ cannot simultaneously be 4 and 8 , then $x$ or $y$ is zero.
- From the constraint $\frac{1}{4} x^{2}+y^{2}=4, x=0 \Longrightarrow y= \pm 2$


## Problem 11 - Spring 2008

Use the method of Lagrange multipliers to determine all points $(x, y)$ where the function $f(x, y)=2 x^{2}+4 y^{2}+16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{1}{4} x^{2}+y^{2}=4$.

## Solution:

- Set $g(x, y)=\frac{1}{4} x^{2}+y^{2}$.
- Set $\nabla f=\langle 4 x, 8 y\rangle=\lambda \nabla g=\lambda\left\langle\frac{1}{2} x, 2 y\right\rangle$ and solve:

$$
\begin{aligned}
& 8 y=2 \lambda y \Longrightarrow \lambda=4 \text { or } y=0 . \\
& 4 x=\frac{1}{2} \lambda x \Longrightarrow \lambda=8 \text { or } x=0 .
\end{aligned}
$$

- Since $\lambda$ cannot simultaneously be 4 and 8 , then $x$ or $y$ is zero.
- From the constraint $\frac{1}{4} x^{2}+y^{2}=4, x=0 \Longrightarrow y= \pm 2$ and $y=0 \Longrightarrow x= \pm 4$.


## Problem 11-Spring 2008

Use the method of Lagrange multipliers to determine all points $(x, y)$ where the function $f(x, y)=2 x^{2}+4 y^{2}+16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{1}{4} x^{2}+y^{2}=4$.

## Solution:

- Set $g(x, y)=\frac{1}{4} x^{2}+y^{2}$.
- Set $\nabla f=\langle 4 x, 8 y\rangle=\lambda \nabla g=\lambda\left\langle\frac{1}{2} x, 2 y\right\rangle$ and solve:

$$
\begin{aligned}
& 8 y=2 \lambda y \Longrightarrow \lambda=4 \text { or } y=0 . \\
& 4 x=\frac{1}{2} \lambda x \Longrightarrow \lambda=8 \text { or } x=0 .
\end{aligned}
$$

- Since $\lambda$ cannot simultaneously be 4 and 8 , then $x$ or $y$ is zero.
- From the constraint $\frac{1}{4} x^{2}+y^{2}=4, x=0 \Longrightarrow y= \pm 2$ and

$$
y=0 \Longrightarrow x= \pm 4 .
$$

- We need to check the values of $f$ at the points $(0, \pm 2),( \pm 4,0)$ :


## Problem 11 - Spring 2008

Use the method of Lagrange multipliers to determine all points $(x, y)$ where the function $f(x, y)=2 x^{2}+4 y^{2}+16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{1}{4} x^{2}+y^{2}=4$.

## Solution:

- Set $g(x, y)=\frac{1}{4} x^{2}+y^{2}$.
- Set $\nabla f=\langle 4 x, 8 y\rangle=\lambda \nabla g=\lambda\left\langle\frac{1}{2} x, 2 y\right\rangle$ and solve:

$$
\begin{aligned}
& 8 y=2 \lambda y \Longrightarrow \lambda=4 \text { or } y=0 . \\
& 4 x=\frac{1}{2} \lambda x \Longrightarrow \lambda=8 \text { or } x=0 .
\end{aligned}
$$

- Since $\lambda$ cannot simultaneously be 4 and 8 , then $x$ or $y$ is zero.
- From the constraint $\frac{1}{4} x^{2}+y^{2}=4, x=0 \Longrightarrow y= \pm 2$ and

$$
y=0 \Longrightarrow x= \pm 4 .
$$

- We need to check the values of $f$ at the points $(0, \pm 2),( \pm 4,0)$ :

$$
f(0, \pm 2)=32 \quad f( \pm 4,0)=48
$$

## Problem 11 - Spring 2008

Use the method of Lagrange multipliers to determine all points $(x, y)$ where the function $f(x, y)=2 x^{2}+4 y^{2}+16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{1}{4} x^{2}+y^{2}=4$.

## Solution:

- Set $g(x, y)=\frac{1}{4} x^{2}+y^{2}$.
- Set $\nabla f=\langle 4 x, 8 y\rangle=\lambda \nabla g=\lambda\left\langle\frac{1}{2} x, 2 y\right\rangle$ and solve:

$$
\begin{aligned}
& 8 y=2 \lambda y \Longrightarrow \lambda=4 \text { or } y=0 . \\
& 4 x=\frac{1}{2} \lambda x \Longrightarrow \lambda=8 \text { or } x=0 .
\end{aligned}
$$

- Since $\lambda$ cannot simultaneously be 4 and 8 , then $x$ or $y$ is zero.
- From the constraint $\frac{1}{4} x^{2}+y^{2}=4, x=0 \Longrightarrow y= \pm 2$ and $y=0 \Longrightarrow x= \pm 4$.
- We need to check the values of $f$ at the points $(0, \pm 2),( \pm 4,0)$ :

$$
f(0, \pm 2)=32 \quad f( \pm 4,0)=48
$$

- Hence, $f(x, y)$ has its minimum value of 32 at the points $(0, \pm 2)$ and its maximum value of 48 at the points $( \pm 4,0)$.


## Problem 12 - Fall 2007

Find the $x$ and $y$ coordinates of all critical points of the function

$$
f(x, y)=2 x^{3}-6 x^{2}+x y^{2}+y^{2}
$$

and use the Second Derivative Test to classify them as local minima, local maxima or saddle points.

## Problem 12 - Fall 2007

Find the $x$ and $y$ coordinates of all critical points of the function

$$
f(x, y)=2 x^{3}-6 x^{2}+x y^{2}+y^{2}
$$

and use the Second Derivative Test to classify them as local minima, local maxima or saddle points.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $(0,0)$ :


## Problem 12 - Fall 2007

Find the $x$ and $y$ coordinates of all critical points of the function

$$
f(x, y)=2 x^{3}-6 x^{2}+x y^{2}+y^{2}
$$

and use the Second Derivative Test to classify them as local minima, local maxima or saddle points.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $(0,0)$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}-12 x+y^{2}, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

## Problem 12 - Fall 2007

Find the $x$ and $y$ coordinates of all critical points of the function

$$
f(x, y)=2 x^{3}-6 x^{2}+x y^{2}+y^{2}
$$

and use the Second Derivative Test to classify them as local minima, local maxima or saddle points.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $(0,0)$ :

$$
\begin{aligned}
& \quad \nabla f(x, y)=\left\langle 6 x^{2}-12 x+y^{2}, 2 x y+2 y\right\rangle=\langle 0,0\rangle . \\
& \Longrightarrow 2 y(x+1)=0
\end{aligned}
$$

## Problem 12 - Fall 2007

Find the $x$ and $y$ coordinates of all critical points of the function

$$
f(x, y)=2 x^{3}-6 x^{2}+x y^{2}+y^{2}
$$

and use the Second Derivative Test to classify them as local minima, local maxima or saddle points.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $(0,0)$ :

$$
\begin{aligned}
& \nabla f(x, y)=\left\langle 6 x^{2}-12 x+y^{2}, 2 x y+2 y\right\rangle=\langle 0,0\rangle . \\
& \Longrightarrow 2 y(x+1)=0 \text { and so } y=0 \text { or } x=-1 .
\end{aligned}
$$

## Problem 12 - Fall 2007

Find the $x$ and $y$ coordinates of all critical points of the function

$$
f(x, y)=2 x^{3}-6 x^{2}+x y^{2}+y^{2}
$$

and use the Second Derivative Test to classify them as local minima, local maxima or saddle points.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $(0,0)$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}-12 x+y^{2}, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

$\Longrightarrow 2 y(x+1)=0$ and so $y=0$ or $x=-1$.

- Suppose $y=0$. Then $6 x^{2}-12 x=6 x(x-2)=0$ and $x=0$ or $x=2$.


## Problem 12 - Fall 2007

Find the $x$ and $y$ coordinates of all critical points of the function

$$
f(x, y)=2 x^{3}-6 x^{2}+x y^{2}+y^{2}
$$

and use the Second Derivative Test to classify them as local minima, local maxima or saddle points.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $(0,0)$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}-12 x+y^{2}, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

$\Longrightarrow 2 y(x+1)=0$ and so $y=0$ or $x=-1$.

- Suppose $y=0$. Then $6 x^{2}-12 x=6 x(x-2)=0$ and $x=0$ or $x=2$.
- Suppose $x=-1$. Then $6+12+y^{2}=0$, which is impossible.


## Problem 12 - Fall 2007

Find the $x$ and $y$ coordinates of all critical points of the function

$$
f(x, y)=2 x^{3}-6 x^{2}+x y^{2}+y^{2}
$$

and use the Second Derivative Test to classify them as local minima, local maxima or saddle points.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $(0,0)$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}-12 x+y^{2}, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

$\Longrightarrow 2 y(x+1)=0$ and so $y=0$ or $x=-1$.

- Suppose $y=0$. Then $6 x^{2}-12 x=6 x(x-2)=0$ and $x=0$ or $x=2$.
- Suppose $x=-1$. Then $6+12+y^{2}=0$, which is impossible.
- The set of critical points is $\{(0,0),(2,0)\}$.


## Problem 12 - Fall 2007

Find the $x$ and $y$ coordinates of all critical points of the function

$$
f(x, y)=2 x^{3}-6 x^{2}+x y^{2}+y^{2}
$$

and use the Second Derivative Test to classify them as local minima, local maxima or saddle points.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $(0,0)$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}-12 x+y^{2}, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

$\Longrightarrow 2 y(x+1)=0$ and so $y=0$ or $x=-1$.

- Suppose $y=0$. Then $6 x^{2}-12 x=6 x(x-2)=0$ and $x=0$ or $x=2$.
- Suppose $x=-1$. Then $6+12+y^{2}=0$, which is impossible.
- The set of critical points is $\{(0,0),(2,0)\}$.
- Next calculate:

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|
$$

## Problem 12 - Fall 2007

Find the $x$ and $y$ coordinates of all critical points of the function

$$
f(x, y)=2 x^{3}-6 x^{2}+x y^{2}+y^{2}
$$

and use the Second Derivative Test to classify them as local minima, local maxima or saddle points.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $(0,0)$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}-12 x+y^{2}, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

$\Longrightarrow 2 y(x+1)=0$ and so $y=0$ or $x=-1$.

- Suppose $y=0$. Then $6 x^{2}-12 x=6 x(x-2)=0$ and $x=0$ or $x=2$.
- Suppose $x=-1$. Then $6+12+y^{2}=0$, which is impossible.
- The set of critical points is $\{(0,0),(2,0)\}$.
- Next calculate:

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x-12 & 2 y \\
2 y & 2 x+2
\end{array}\right|
$$

## Problem 12 - Fall 2007

Find the $x$ and $y$ coordinates of all critical points of the function

$$
f(x, y)=2 x^{3}-6 x^{2}+x y^{2}+y^{2}
$$

and use the Second Derivative Test to classify them as local minima, local maxima or saddle points.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $(0,0)$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}-12 x+y^{2}, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

$\Longrightarrow 2 y(x+1)=0$ and so $y=0$ or $x=-1$.

- Suppose $y=0$. Then $6 x^{2}-12 x=6 x(x-2)=0$ and $x=0$ or $x=2$.
- Suppose $x=-1$. Then $6+12+y^{2}=0$, which is impossible.
- The set of critical points is $\{(0,0),(2,0)\}$.
- Next calculate:

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x-12 & 2 y \\
2 y & 2 x+2
\end{array}\right|=24 x^{2}-24-4 y^{2} .
$$

## Problem 12-Fall 2007

Find the $x$ and $y$ coordinates of all critical points of the function

$$
f(x, y)=2 x^{3}-6 x^{2}+x y^{2}+y^{2}
$$

and use the Second Derivative Test to classify them as local minima, local maxima or saddle points.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $(0,0)$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}-12 x+y^{2}, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

$\Longrightarrow 2 y(x+1)=0$ and so $y=0$ or $x=-1$.

- Suppose $y=0$. Then $6 x^{2}-12 x=6 x(x-2)=0$ and $x=0$ or $x=2$.
- Suppose $x=-1$. Then $6+12+y^{2}=0$, which is impossible.
- The set of critical points is $\{(0,0),(2,0)\}$.
- Next calculate:

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x-12 & 2 y \\
2 y & 2 x+2
\end{array}\right|=24 x^{2}-24-4 y^{2} .
$$

- $D(0,0)=-24<0$ and so $(0,0)$ is a saddle point.


## Problem 12-Fall 2007

Find the $x$ and $y$ coordinates of all critical points of the function

$$
f(x, y)=2 x^{3}-6 x^{2}+x y^{2}+y^{2}
$$

and use the Second Derivative Test to classify them as local minima, local maxima or saddle points.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $(0,0)$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}-12 x+y^{2}, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

$\Longrightarrow 2 y(x+1)=0$ and so $y=0$ or $x=-1$.

- Suppose $y=0$. Then $6 x^{2}-12 x=6 x(x-2)=0$ and $x=0$ or $x=2$.
- Suppose $x=-1$. Then $6+12+y^{2}=0$, which is impossible.
- The set of critical points is $\{(0,0),(2,0)\}$.
- Next calculate:

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x-12 & 2 y \\
2 y & 2 x+2
\end{array}\right|=24 x^{2}-24-4 y^{2}
$$

- $D(0,0)=-24<0$ and so $(0,0)$ is a saddle point.
- $D(2,0)=96-24=72>0$ and $f_{x x}(2,0)=12>0$,


## Problem 12-Fall 2007

Find the $x$ and $y$ coordinates of all critical points of the function

$$
f(x, y)=2 x^{3}-6 x^{2}+x y^{2}+y^{2}
$$

and use the Second Derivative Test to classify them as local minima, local maxima or saddle points.

## Solution:

- First calculate $\nabla f(x, y)$ and set equal to $(0,0)$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}-12 x+y^{2}, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

$\Longrightarrow 2 y(x+1)=0$ and so $y=0$ or $x=-1$.

- Suppose $y=0$. Then $6 x^{2}-12 x=6 x(x-2)=0$ and $x=0$ or $x=2$.
- Suppose $x=-1$. Then $6+12+y^{2}=0$, which is impossible.
- The set of critical points is $\{(0,0),(2,0)\}$.
- Next calculate:

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x-12 & 2 y \\
2 y & 2 x+2
\end{array}\right|=24 x^{2}-24-4 y^{2} .
$$

- $D(0,0)=-24<0$ and so $(0,0)$ is a saddle point.
- $D(2,0)=96-24=72>0$ and $f_{x x}(2,0)=12>0$, so $(2,0)$ is a local minimum.


## Problem 13(a) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=1-4 x^{2}-3 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P\left(\frac{1}{4},-\frac{1}{2}, 0\right)$ heading North, is she ascending or descending? Justify your answers.

## Problem 13(a) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=1-4 x^{2}-3 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P\left(\frac{1}{4},-\frac{1}{2}, 0\right)$ heading North, is she ascending or descending? Justify your answers.

## Solution:

- Let $f(x, y)=z=1-4 x^{2}-3 y^{2}$.


## Problem 13(a) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=1-4 x^{2}-3 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P\left(\frac{1}{4},-\frac{1}{2}, 0\right)$ heading North, is she ascending or descending? Justify your answers.

## Solution:

- Let $f(x, y)=z=1-4 x^{2}-3 y^{2}$.
- This is a problem where we need to calculate the sign of the directional derivative $D_{\langle 0,1\rangle} f\left(\frac{1}{4},-\frac{1}{2}\right)=\nabla f\left(\frac{1}{4},-\frac{1}{2}\right) \cdot\langle 0,1\rangle$, where $\langle 0,1\rangle$ represents North.


## Problem 13(a) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=1-4 x^{2}-3 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P\left(\frac{1}{4},-\frac{1}{2}, 0\right)$ heading North, is she ascending or descending? Justify your answers.

## Solution:

- Let $f(x, y)=z=1-4 x^{2}-3 y^{2}$.
- This is a problem where we need to calculate the sign of the directional derivative $D_{\langle 0,1\rangle} f\left(\frac{1}{4},-\frac{1}{2}\right)=\nabla f\left(\frac{1}{4},-\frac{1}{2}\right) \cdot\langle 0,1\rangle$, where $\langle 0,1\rangle$ represents North.
- Calculating, we obtain:

$$
\nabla f(x, y)=\langle-8 x,-6 y\rangle \quad \nabla f\left(\frac{1}{4},-\frac{1}{2}\right)=\langle-2,3\rangle .
$$

## Problem 13(a) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=1-4 x^{2}-3 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P\left(\frac{1}{4},-\frac{1}{2}, 0\right)$ heading North, is she ascending or descending? Justify your answers.

## Solution:

- Let $f(x, y)=z=1-4 x^{2}-3 y^{2}$.
- This is a problem where we need to calculate the sign of the directional derivative $D_{\langle 0,1\rangle} f\left(\frac{1}{4},-\frac{1}{2}\right)=\nabla f\left(\frac{1}{4},-\frac{1}{2}\right) \cdot\langle 0,1\rangle$, where $\langle 0,1\rangle$ represents North.
- Calculating, we obtain:

$$
\nabla f(x, y)=\langle-8 x,-6 y\rangle \quad \nabla f\left(\frac{1}{4},-\frac{1}{2}\right)=\langle-2,3\rangle .
$$

- Hence,

$$
D_{\langle 0,1\rangle} f\left(\frac{1}{4},-\frac{1}{2}\right)
$$

## Problem 13(a) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=1-4 x^{2}-3 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P\left(\frac{1}{4},-\frac{1}{2}, 0\right)$ heading North, is she ascending or descending? Justify your answers.

## Solution:

- Let $f(x, y)=z=1-4 x^{2}-3 y^{2}$.
- This is a problem where we need to calculate the sign of the directional derivative $D_{\langle 0,1\rangle} f\left(\frac{1}{4},-\frac{1}{2}\right)=\nabla f\left(\frac{1}{4},-\frac{1}{2}\right) \cdot\langle 0,1\rangle$, where $\langle 0,1\rangle$ represents North.
- Calculating, we obtain:

$$
\nabla f(x, y)=\langle-8 x,-6 y\rangle \quad \nabla f\left(\frac{1}{4},-\frac{1}{2}\right)=\langle-2,3\rangle
$$

- Hence,

$$
D_{\langle 0,1\rangle} f\left(\frac{1}{4},-\frac{1}{2}\right)=\langle-2,3\rangle \cdot\langle 0,1\rangle
$$

## Problem 13(a) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=1-4 x^{2}-3 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P\left(\frac{1}{4},-\frac{1}{2}, 0\right)$ heading North, is she ascending or descending? Justify your answers.

## Solution:

- Let $f(x, y)=z=1-4 x^{2}-3 y^{2}$.
- This is a problem where we need to calculate the sign of the directional derivative $D_{\langle 0,1\rangle} f\left(\frac{1}{4},-\frac{1}{2}\right)=\nabla f\left(\frac{1}{4},-\frac{1}{2}\right) \cdot\langle 0,1\rangle$, where $\langle 0,1\rangle$ represents North.
- Calculating, we obtain:

$$
\nabla f(x, y)=\langle-8 x,-6 y\rangle \quad \nabla f\left(\frac{1}{4},-\frac{1}{2}\right)=\langle-2,3\rangle
$$

- Hence,

$$
D_{\langle 0,1\rangle} f\left(\frac{1}{4},-\frac{1}{2}\right)=\langle-2,3\rangle \cdot\langle 0,1\rangle=3>0,
$$

## Problem 13(a) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=1-4 x^{2}-3 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Suppose the hiker is now at the point $P\left(\frac{1}{4},-\frac{1}{2}, 0\right)$ heading North, is she ascending or descending? Justify your answers.

## Solution:

- Let $f(x, y)=z=1-4 x^{2}-3 y^{2}$.
- This is a problem where we need to calculate the sign of the directional derivative $D_{\langle 0,1\rangle} f\left(\frac{1}{4},-\frac{1}{2}\right)=\nabla f\left(\frac{1}{4},-\frac{1}{2}\right) \cdot\langle 0,1\rangle$, where $\langle 0,1\rangle$ represents North.
- Calculating, we obtain:

$$
\nabla f(x, y)=\langle-8 x,-6 y\rangle \quad \nabla f\left(\frac{1}{4},-\frac{1}{2}\right)=\langle-2,3\rangle
$$

- Hence,

$$
D_{\langle 0,1\rangle} f\left(\frac{1}{4},-\frac{1}{2}\right)=\langle-2,3\rangle \cdot\langle 0,1\rangle=3>0
$$

which means that she is ascending.

## Problem 13(b) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=1-4 x^{2}-3 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Justify your answers.
When the hiker is at the point $Q\left(\frac{1}{4}, 0, \frac{3}{4}\right)$, in which direction should she initially head to ascend most rapidly?

## Problem 13(b) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=1-4 x^{2}-3 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Justify your answers.
When the hiker is at the point $Q\left(\frac{1}{4}, 0, \frac{3}{4}\right)$, in which direction should she initially head to ascend most rapidly?

## Solution:

- Recall that $\nabla f(x, y)=\langle-8 x,-6 y\rangle$.


## Problem 13(b) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=1-4 x^{2}-3 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Justify your answers.
When the hiker is at the point $Q\left(\frac{1}{4}, 0, \frac{3}{4}\right)$, in which direction should she initially head to ascend most rapidly?

## Solution:

- Recall that $\nabla f(x, y)=\langle-8 x,-6 y\rangle$.
- The direction of greatest ascent is in the direction $\mathbf{v}=\nabla f$ at the point $\left(\frac{1}{4}, 0\right)$ in the $x y$-plane.


## Problem 13(b) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=1-4 x^{2}-3 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Justify your answers.
When the hiker is at the point $Q\left(\frac{1}{4}, 0, \frac{3}{4}\right)$, in which direction should she initially head to ascend most rapidly?

## Solution:

- Recall that $\nabla f(x, y)=\langle-8 x,-6 y\rangle$.
- The direction of greatest ascent is in the direction $\mathbf{v}=\nabla f$ at the point $\left(\frac{1}{4}, 0\right)$ in the $x y$-plane.
- Thus,

$$
\mathbf{v}=\nabla f\left(\frac{1}{4}, 0\right)
$$

## Problem 13(b) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=1-4 x^{2}-3 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Justify your answers.
When the hiker is at the point $Q\left(\frac{1}{4}, 0, \frac{3}{4}\right)$, in which direction should she initially head to ascend most rapidly?

## Solution:

- Recall that $\nabla f(x, y)=\langle-8 x,-6 y\rangle$.
- The direction of greatest ascent is in the direction $\mathbf{v}=\nabla f$ at the point $\left(\frac{1}{4}, 0\right)$ in the $x y$-plane.
- Thus,

$$
\mathbf{v}=\nabla f\left(\frac{1}{4}, 0\right)=\langle-2,0\rangle
$$

## Problem 13(b) - Fall 2007

A hiker is walking on a mountain path. The surface of the mountain is modeled by $z=1-4 x^{2}-3 y^{2}$. The positive $x$-axis points to East direction and the positive $y$-axis points North. Justify your answers.
When the hiker is at the point $Q\left(\frac{1}{4}, 0, \frac{3}{4}\right)$, in which direction should she initially head to ascend most rapidly?

## Solution:

- Recall that $\nabla f(x, y)=\langle-8 x,-6 y\rangle$.
- The direction of greatest ascent is in the direction $\mathbf{v}=\nabla f$ at the point $\left(\frac{1}{4}, 0\right)$ in the $x y$-plane.
- Thus,

$$
\mathbf{v}=\nabla f\left(\frac{1}{4}, 0\right)=\langle-2,0\rangle
$$

which means that she should go West.

## Problem 14 - Fall 2007

Find the volume $\mathbf{V}$ of the solid bounded by the surface $z=6-x y$ and the planes $x=2, x=-2, y=0, y=3$ and $z=0$.

## Problem 14 - Fall 2007

Find the volume $\mathbf{V}$ of the solid bounded by the surface $z=6-x y$ and the planes $x=2, x=-2, y=0, y=3$ and $z=0$.

## Solution:

- Note that the graph of $f(x, y)=z=6-x y$ is nonnegative over the rectangle $\mathbf{R}=[-2,2] \times[0,3]$ and the volume $\mathbf{V}$ described is the volume under the graph.


## Problem 14 - Fall 2007

Find the volume $\mathbf{V}$ of the solid bounded by the surface $z=6-x y$ and the planes $x=2, x=-2, y=0, y=3$ and $z=0$.

## Solution:

- Note that the graph of $f(x, y)=z=6-x y$ is nonnegative over the rectangle $\mathbf{R}=[-2,2] \times[0,3]$ and the volume $\mathbf{V}$ described is the volume under the graph.
- Applying Fubini's Theorem gives:

$$
\mathbf{V}=\int_{-2}^{2} \int_{0}^{3} 6-x y d y d x
$$

## Problem 14 - Fall 2007

Find the volume $\mathbf{V}$ of the solid bounded by the surface $z=6-x y$ and the planes $x=2, x=-2, y=0, y=3$ and $z=0$.

## Solution:

- Note that the graph of $f(x, y)=z=6-x y$ is nonnegative over the rectangle $\mathbf{R}=[-2,2] \times[0,3]$ and the volume $\mathbf{V}$ described is the volume under the graph.
- Applying Fubini's Theorem gives:

$$
\mathbf{V}=\int_{-2}^{2} \int_{0}^{3} 6-x y d y d x=\int_{-2}^{2}\left[6 y-\frac{1}{2} x y^{2}\right]_{0}^{3} d x
$$

## Problem 14 - Fall 2007

Find the volume $\mathbf{V}$ of the solid bounded by the surface $z=6-x y$ and the planes $x=2, x=-2, y=0, y=3$ and $z=0$.

## Solution:

- Note that the graph of $f(x, y)=z=6-x y$ is nonnegative over the rectangle $\mathbf{R}=[-2,2] \times[0,3]$ and the volume $\mathbf{V}$ described is the volume under the graph.
- Applying Fubini's Theorem gives:

$$
\begin{aligned}
& \quad \mathbf{V}=\int_{-2}^{2} \int_{0}^{3} 6-x y d y d x=\int_{-2}^{2}\left[6 y-\frac{1}{2} x y^{2}\right]_{0}^{3} d x \\
& =\int_{-2}^{2}\left(18-\frac{9}{2} x\right) d x
\end{aligned}
$$

## Problem 14 - Fall 2007

Find the volume $\mathbf{V}$ of the solid bounded by the surface $z=6-x y$ and the planes $x=2, x=-2, y=0, y=3$ and $z=0$.

## Solution:

- Note that the graph of $f(x, y)=z=6-x y$ is nonnegative over the rectangle $\mathbf{R}=[-2,2] \times[0,3]$ and the volume $\mathbf{V}$ described is the volume under the graph.
- Applying Fubini's Theorem gives:

$$
\begin{aligned}
& \quad \mathbf{V}=\int_{-2}^{2} \int_{0}^{3} 6-x y d y d x=\int_{-2}^{2}\left[6 y-\frac{1}{2} x y^{2}\right]_{0}^{3} d x \\
& =\int_{-2}^{2}\left(18-\frac{9}{2} x\right) d x=18 x-\left.\frac{9}{4} x^{2}\right|_{-2} ^{2}
\end{aligned}
$$

## Problem 14 - Fall 2007

Find the volume $\mathbf{V}$ of the solid bounded by the surface $z=6-x y$ and the planes $x=2, x=-2, y=0, y=3$ and $z=0$.

## Solution:

- Note that the graph of $f(x, y)=z=6-x y$ is nonnegative over the rectangle $\mathbf{R}=[-2,2] \times[0,3]$ and the volume $\mathbf{V}$ described is the volume under the graph.
- Applying Fubini's Theorem gives:

$$
\begin{gathered}
\mathbf{V}=\int_{-2}^{2} \int_{0}^{3} 6-x y d y d x=\int_{-2}^{2}\left[6 y-\frac{1}{2} x y^{2}\right]_{0}^{3} d x \\
=\int_{-2}^{2}\left(18-\frac{9}{2} x\right) d x=18 x-\left.\frac{9}{4} x^{2}\right|_{-2} ^{2}=(36-9)-(-36-9)
\end{gathered}
$$

## Problem 14 - Fall 2007

Find the volume $\mathbf{V}$ of the solid bounded by the surface $z=6-x y$ and the planes $x=2, x=-2, y=0, y=3$ and $z=0$.

## Solution:

- Note that the graph of $f(x, y)=z=6-x y$ is nonnegative over the rectangle $\mathbf{R}=[-2,2] \times[0,3]$ and the volume $\mathbf{V}$ described is the volume under the graph.
- Applying Fubini's Theorem gives:

$$
\begin{gathered}
\mathbf{V}=\int_{-2}^{2} \int_{0}^{3} 6-x y d y d x=\int_{-2}^{2}\left[6 y-\frac{1}{2} x y^{2}\right]_{0}^{3} d x \\
=\int_{-2}^{2}\left(18-\frac{9}{2} x\right) d x=18 x-\left.\frac{9}{4} x^{2}\right|_{-2} ^{2}=(36-9)-(-36-9)=72 .
\end{gathered}
$$

## Problem 15 - Fall 2007

Let $z(x, y)=x^{2}+y^{2}-x y$ where $x=s-r$ and $y=y(r, s)$ is an unknown function of $r$ and $s$. (Note that $z$ can be considered a function of $r$ and $s$.) Suppose we know that

$$
y(2,3)=3, \quad \frac{\partial y}{\partial r}(2,3)=7, \quad \text { and } \frac{\partial y}{\partial s}(2,3)=-5 .
$$

Calculate $\frac{\partial z}{\partial r}$ when $r=2$ and $s=3$.

## Problem 15 - Fall 2007

Let $z(x, y)=x^{2}+y^{2}-x y$ where $x=s-r$ and $y=y(r, s)$ is an unknown function of $r$ and $s$. (Note that $z$ can be considered a function of $r$ and $s$.) Suppose we know that

$$
y(2,3)=3, \quad \frac{\partial y}{\partial r}(2,3)=7, \quad \text { and } \frac{\partial y}{\partial s}(2,3)=-5 .
$$

Calculate $\frac{\partial z}{\partial r}$ when $r=2$ and $s=3$.

## Solution:

- By the Chain Rule:

$$
\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial r}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial r}
$$

## Problem 15 - Fall 2007

Let $z(x, y)=x^{2}+y^{2}-x y$ where $x=s-r$ and $y=y(r, s)$ is an unknown function of $r$ and $s$. (Note that $z$ can be considered a function of $r$ and $s$.) Suppose we know that

$$
y(2,3)=3, \quad \frac{\partial y}{\partial r}(2,3)=7, \quad \text { and } \frac{\partial y}{\partial s}(2,3)=-5 .
$$

Calculate $\frac{\partial z}{\partial r}$ when $r=2$ and $s=3$.

## Solution:

- By the Chain Rule:

$$
\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial r}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial r}=(2 x-y) \frac{\partial x}{\partial r}+(2 y-x) \frac{\partial y}{\partial r}
$$

## Problem 15 - Fall 2007

Let $z(x, y)=x^{2}+y^{2}-x y$ where $x=s-r$ and $y=y(r, s)$ is an unknown function of $r$ and $s$. (Note that $z$ can be considered a function of $r$ and $s$.) Suppose we know that

$$
y(2,3)=3, \quad \frac{\partial y}{\partial r}(2,3)=7, \quad \text { and } \frac{\partial y}{\partial s}(2,3)=-5 .
$$

Calculate $\frac{\partial z}{\partial r}$ when $r=2$ and $s=3$.

## Solution:

- By the Chain Rule:

$$
\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial r}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial r}=(2 x-y) \frac{\partial x}{\partial r}+(2 y-x) \frac{\partial y}{\partial r}
$$

- Note that $r=2$ and $s=3 \Longrightarrow x=1$ and $y=3$.


## Problem 15-Fall 2007

Let $z(x, y)=x^{2}+y^{2}-x y$ where $x=s-r$ and $y=y(r, s)$ is an unknown function of $r$ and $s$. (Note that $z$ can be considered a function of $r$ and $s$.) Suppose we know that

$$
y(2,3)=3, \quad \frac{\partial y}{\partial r}(2,3)=7, \quad \text { and } \frac{\partial y}{\partial s}(2,3)=-5 .
$$

Calculate $\frac{\partial z}{\partial r}$ when $r=2$ and $s=3$.

## Solution:

- By the Chain Rule:

$$
\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial r}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial r}=(2 x-y) \frac{\partial x}{\partial r}+(2 y-x) \frac{\partial y}{\partial r}
$$

- Note that $r=2$ and $s=3 \Longrightarrow x=1$ and $y=3$.
- Hence,

$$
\frac{\partial z}{\partial r}
$$

## Problem 15-Fall 2007

Let $z(x, y)=x^{2}+y^{2}-x y$ where $x=s-r$ and $y=y(r, s)$ is an unknown function of $r$ and $s$. (Note that $z$ can be considered a function of $r$ and $s$.) Suppose we know that

$$
y(2,3)=3, \quad \frac{\partial y}{\partial r}(2,3)=7, \quad \text { and } \frac{\partial y}{\partial s}(2,3)=-5 .
$$

Calculate $\frac{\partial z}{\partial r}$ when $r=2$ and $s=3$.

## Solution:

- By the Chain Rule:

$$
\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial r}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial r}=(2 x-y) \frac{\partial x}{\partial r}+(2 y-x) \frac{\partial y}{\partial r}
$$

- Note that $r=2$ and $s=3 \Longrightarrow x=1$ and $y=3$.
- Hence,

$$
\frac{\partial z}{\partial r}=(2-3)(-1)+(6-1) 7
$$

## Problem 15-Fall 2007

Let $z(x, y)=x^{2}+y^{2}-x y$ where $x=s-r$ and $y=y(r, s)$ is an unknown function of $r$ and $s$. (Note that $z$ can be considered a function of $r$ and $s$.) Suppose we know that

$$
y(2,3)=3, \quad \frac{\partial y}{\partial r}(2,3)=7, \quad \text { and } \frac{\partial y}{\partial s}(2,3)=-5 .
$$

Calculate $\frac{\partial z}{\partial r}$ when $r=2$ and $s=3$.

## Solution:

- By the Chain Rule:

$$
\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial r}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial r}=(2 x-y) \frac{\partial x}{\partial r}+(2 y-x) \frac{\partial y}{\partial r}
$$

- Note that $r=2$ and $s=3 \Longrightarrow x=1$ and $y=3$.
- Hence,

$$
\frac{\partial z}{\partial r}=(2-3)(-1)+(6-1) 7=1+35
$$

## Problem 15-Fall 2007

Let $z(x, y)=x^{2}+y^{2}-x y$ where $x=s-r$ and $y=y(r, s)$ is an unknown function of $r$ and $s$. (Note that $z$ can be considered a function of $r$ and $s$.) Suppose we know that

$$
y(2,3)=3, \quad \frac{\partial y}{\partial r}(2,3)=7, \quad \text { and } \frac{\partial y}{\partial s}(2,3)=-5 .
$$

Calculate $\frac{\partial z}{\partial r}$ when $r=2$ and $s=3$.

## Solution:

- By the Chain Rule:

$$
\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial r}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial r}=(2 x-y) \frac{\partial x}{\partial r}+(2 y-x) \frac{\partial y}{\partial r} .
$$

- Note that $r=2$ and $s=3 \Longrightarrow x=1$ and $y=3$.
- Hence,

$$
\frac{\partial z}{\partial r}=(2-3)(-1)+(6-1) 7=1+35=36
$$

Problem 16(a) - Fall 2007
Let $\mathbf{F}(x, y, z)=x^{2}-2 x y-y^{2}+8 x+4 y-z$.
Write the equation of the tangent plane to the surface given by $\mathbf{F}(x, y, z)=0$ at the point $(-2,1,-5)$.

Problem 16(a) - Fall 2007
Let $\mathbf{F}(x, y, z)=x^{2}-2 x y-y^{2}+8 x+4 y-z$.
Write the equation of the tangent plane to the surface given by $\mathbf{F}(x, y, z)=0$ at the point $(-2,1,-5)$.

## Solution:

- Note that the normal $\mathbf{n}$ of the plane is $\nabla \mathbf{F}(-2,1,-5)$.


## Problem 16(a) - Fall 2007

Let $\mathbf{F}(x, y, z)=x^{2}-2 x y-y^{2}+8 x+4 y-z$.
Write the equation of the tangent plane to the surface given by $\mathbf{F}(x, y, z)=0$ at the point $(-2,1,-5)$.

## Solution:

- Note that the normal $\mathbf{n}$ of the plane is $\nabla \mathbf{F}(-2,1,-5)$.
- Calculating, we obtain:

$$
\nabla \mathbf{F}(x, y, z)=\langle 2 x-2 y+8,-2 x-2 y+4,-1\rangle
$$

## Problem 16(a) - Fall 2007

Let $\mathbf{F}(x, y, z)=x^{2}-2 x y-y^{2}+8 x+4 y-z$.
Write the equation of the tangent plane to the surface given by $\mathbf{F}(x, y, z)=0$ at the point $(-2,1,-5)$.

## Solution:

- Note that the normal $\mathbf{n}$ of the plane is $\nabla \mathbf{F}(-2,1,-5)$.
- Calculating, we obtain:

$$
\nabla \mathbf{F}(x, y, z)=\langle 2 x-2 y+8,-2 x-2 y+4,-1\rangle
$$

- So,

$$
\mathbf{n}=\nabla \mathbf{F}(-2,1,-5)=\langle-4-2+8,4-2+4,-1\rangle=\langle 2,6,-1\rangle .
$$

## Problem 16(a) - Fall 2007

Let $\mathbf{F}(x, y, z)=x^{2}-2 x y-y^{2}+8 x+4 y-z$.
Write the equation of the tangent plane to the surface given by $\mathbf{F}(x, y, z)=0$ at the point $(-2,1,-5)$.

## Solution:

- Note that the normal $\mathbf{n}$ of the plane is $\nabla \mathbf{F}(-2,1,-5)$.
- Calculating, we obtain:

$$
\nabla \mathbf{F}(x, y, z)=\langle 2 x-2 y+8,-2 x-2 y+4,-1\rangle
$$

- So,

$$
\mathbf{n}=\nabla \mathbf{F}(-2,1,-5)=\langle-4-2+8,4-2+4,-1\rangle=\langle 2,6,-1\rangle .
$$

- The equation of the tangent plane is:

$$
\mathbf{n} \cdot\langle x+2, y-1, z+5\rangle=2(x+2)+6(y-1)-(z+5)=0 .
$$

## Problem 16(b) - Fall 2007

Find the point ( $a, b, c$ ) on the surface $\mathbf{F}(x, y, z)=0$ at which the tangent plane is horizontal, that is, parallel to the $z=0$ plane.

## Problem 16(b) - Fall 2007

Find the point $(a, b, c)$ on the surface $\mathbf{F}(x, y, z)=0$ at which the tangent plane is horizontal, that is, parallel to the $z=0$ plane.

## Solution:

- Since $\nabla \mathbf{F}$ is normal to the surface $\mathbf{F}(x, y, z)=0$, a horizontal tangent plane to the surface occurs where $\nabla \mathrm{F}$ is vertical.


## Problem 16(b) - Fall 2007

Find the point $(a, b, c)$ on the surface $\mathbf{F}(x, y, z)=0$ at which the tangent plane is horizontal, that is, parallel to the $z=0$ plane.

## Solution:

- Since $\nabla \mathbf{F}$ is normal to the surface $\mathbf{F}(x, y, z)=0$, a horizontal tangent plane to the surface occurs where $\nabla \mathbf{F}$ is vertical.
- $\nabla \mathbf{F}$ is vertical on $\mathbf{F}(x, y, z)=0$, when its first 2 coordinates vanish:


## Problem 16(b) - Fall 2007

Find the point $(a, b, c)$ on the surface $\mathbf{F}(x, y, z)=0$ at which the tangent plane is horizontal, that is, parallel to the $z=0$ plane.

## Solution:

- Since $\nabla \mathbf{F}$ is normal to the surface $\mathbf{F}(x, y, z)=0$, a horizontal tangent plane to the surface occurs where $\nabla \mathbf{F}$ is vertical.
- $\nabla \mathbf{F}$ is vertical on $\mathbf{F}(x, y, z)=0$, when its first 2 coordinates vanish:

$$
\nabla \mathbf{F}=\langle 2 x-2 y+8,-2 x-2 y+4,-1\rangle=\langle 0,0,-1\rangle
$$

## Problem 16(b) - Fall 2007

Find the point $(a, b, c)$ on the surface $\mathbf{F}(x, y, z)=0$ at which the tangent plane is horizontal, that is, parallel to the $z=0$ plane.

## Solution:

- Since $\nabla \mathbf{F}$ is normal to the surface $\mathbf{F}(x, y, z)=0$, a horizontal tangent plane to the surface occurs where $\nabla \mathbf{F}$ is vertical.
- $\nabla \mathbf{F}$ is vertical on $\mathbf{F}(x, y, z)=0$, when its first 2 coordinates vanish:

$$
\begin{gathered}
\nabla \mathbf{F}=\langle 2 x-2 y+8,-2 x-2 y+4,-1\rangle=\langle 0,0,-1\rangle \Longrightarrow \\
2 x-2 y+8=0 \\
-2 x-2 y+4=0
\end{gathered}
$$

## Problem 16(b) - Fall 2007

Find the point $(a, b, c)$ on the surface $\mathbf{F}(x, y, z)=0$ at which the tangent plane is horizontal, that is, parallel to the $z=0$ plane.

## Solution:

- Since $\nabla \mathbf{F}$ is normal to the surface $\mathbf{F}(x, y, z)=0$, a horizontal tangent plane to the surface occurs where $\nabla \mathbf{F}$ is vertical.
- $\nabla \mathbf{F}$ is vertical on $\mathbf{F}(x, y, z)=0$, when its first 2 coordinates vanish:

$$
\begin{gathered}
\nabla \mathbf{F}=\langle 2 x-2 y+8,-2 x-2 y+4,-1\rangle=\langle 0,0,-1\rangle \Longrightarrow \\
2 x-2 y+8=0 \\
-2 x-2 y+4=0
\end{gathered}
$$

- Adding these equations $\Longrightarrow 4 y=12$


## Problem 16(b) - Fall 2007

Find the point $(a, b, c)$ on the surface $\mathbf{F}(x, y, z)=0$ at which the tangent plane is horizontal, that is, parallel to the $z=0$ plane.

## Solution:

- Since $\nabla \mathbf{F}$ is normal to the surface $\mathbf{F}(x, y, z)=0$, a horizontal tangent plane to the surface occurs where $\nabla \mathbf{F}$ is vertical.
- $\nabla \mathbf{F}$ is vertical on $\mathbf{F}(x, y, z)=0$, when its first 2 coordinates vanish:

$$
\begin{gathered}
\nabla \mathbf{F}=\langle 2 x-2 y+8,-2 x-2 y+4,-1\rangle=\langle 0,0,-1\rangle \Longrightarrow \\
2 x-2 y+8=0 \\
-2 x-2 y+4=0
\end{gathered}
$$

- Adding these equations $\Longrightarrow 4 y=12 \Longrightarrow y=3$.


## Problem 16(b) - Fall 2007

Find the point $(a, b, c)$ on the surface $\mathbf{F}(x, y, z)=0$ at which the tangent plane is horizontal, that is, parallel to the $z=0$ plane.

## Solution:

- Since $\nabla \mathbf{F}$ is normal to the surface $\mathbf{F}(x, y, z)=0$, a horizontal tangent plane to the surface occurs where $\nabla \mathbf{F}$ is vertical.
- $\nabla \mathbf{F}$ is vertical on $\mathbf{F}(x, y, z)=0$, when its first 2 coordinates vanish:

$$
\begin{gathered}
\nabla \mathbf{F}=\langle 2 x-2 y+8,-2 x-2 y+4,-1\rangle=\langle 0,0,-1\rangle \Longrightarrow \\
2 x-2 y+8=0 \\
-2 x-2 y+4=0
\end{gathered}
$$

- Adding these equations $\Longrightarrow 4 y=12 \Longrightarrow y=3$.
- Plugging in $y=3$ in first equation gives

$$
2 x-2 \cdot 3+8=0
$$

## Problem 16(b) - Fall 2007

Find the point $(a, b, c)$ on the surface $\mathbf{F}(x, y, z)=0$ at which the tangent plane is horizontal, that is, parallel to the $z=0$ plane.

## Solution:

- Since $\nabla \mathbf{F}$ is normal to the surface $\mathbf{F}(x, y, z)=0$, a horizontal tangent plane to the surface occurs where $\nabla \mathbf{F}$ is vertical.
- $\nabla \mathbf{F}$ is vertical on $\mathbf{F}(x, y, z)=0$, when its first 2 coordinates vanish:

$$
\begin{gathered}
\nabla \mathbf{F}=\langle 2 x-2 y+8,-2 x-2 y+4,-1\rangle=\langle 0,0,-1\rangle \Longrightarrow \\
2 x-2 y+8=0 \\
-2 x-2 y+4=0
\end{gathered}
$$

- Adding these equations $\Longrightarrow 4 y=12 \Longrightarrow y=3$.
- Plugging in $y=3$ in first equation gives

$$
2 x-2 \cdot 3+8=0 \Longrightarrow x=-1
$$

## Problem 16(b) - Fall 2007

Find the point $(a, b, c)$ on the surface $\mathbf{F}(x, y, z)=0$ at which the tangent plane is horizontal, that is, parallel to the $z=0$ plane.

## Solution:

- Since $\nabla \mathbf{F}$ is normal to the surface $\mathbf{F}(x, y, z)=0$, a horizontal tangent plane to the surface occurs where $\nabla \mathbf{F}$ is vertical.
- $\nabla \mathbf{F}$ is vertical on $\mathbf{F}(x, y, z)=0$, when its first 2 coordinates vanish:

$$
\begin{gathered}
\nabla \mathbf{F}=\langle 2 x-2 y+8,-2 x-2 y+4,-1\rangle=\langle 0,0,-1\rangle \Longrightarrow \\
2 x-2 y+8=0 \\
-2 x-2 y+4=0
\end{gathered}
$$

- Adding these equations $\Longrightarrow 4 y=12 \Longrightarrow y=3$.
- Plugging in $y=3$ in first equation gives

$$
2 x-2 \cdot 3+8=0 \Longrightarrow x=-1
$$

- $\mathbf{F}(x, y, z)=x^{2}-2 x y-y^{2}+8 x+4 y-z$ and $\mathbf{F}(-1,3, z)=0$, $\Longrightarrow$


## Problem 16(b) - Fall 2007

Find the point $(a, b, c)$ on the surface $\mathbf{F}(x, y, z)=0$ at which the tangent plane is horizontal, that is, parallel to the $z=0$ plane.

## Solution:

- Since $\nabla \mathbf{F}$ is normal to the surface $\mathbf{F}(x, y, z)=0$, a horizontal tangent plane to the surface occurs where $\nabla \mathbf{F}$ is vertical.
- $\nabla \mathbf{F}$ is vertical on $\mathbf{F}(x, y, z)=0$, when its first 2 coordinates vanish:

$$
\begin{gathered}
\nabla \mathbf{F}=\langle 2 x-2 y+8,-2 x-2 y+4,-1\rangle=\langle 0,0,-1\rangle \Longrightarrow \\
2 x-2 y+8=0 \\
-2 x-2 y+4=0
\end{gathered}
$$

- Adding these equations $\Longrightarrow 4 y=12 \Longrightarrow y=3$.
- Plugging in $y=3$ in first equation gives

$$
2 x-2 \cdot 3+8=0 \Longrightarrow x=-1
$$

- $\mathbf{F}(x, y, z)=x^{2}-2 x y-y^{2}+8 x+4 y-z$ and $\mathbf{F}(-1,3, z)=0$,
$\Longrightarrow z=(-1)^{2}-2(-1)(3)-3^{2}+8(-1)+4(3)=2$.


## Problem 16(b) - Fall 2007

Find the point $(a, b, c)$ on the surface $\mathbf{F}(x, y, z)=0$ at which the tangent plane is horizontal, that is, parallel to the $z=0$ plane.

## Solution:

- Since $\nabla \mathbf{F}$ is normal to the surface $\mathbf{F}(x, y, z)=0$, a horizontal tangent plane to the surface occurs where $\nabla \mathbf{F}$ is vertical.
- $\nabla \mathbf{F}$ is vertical on $\mathbf{F}(x, y, z)=0$, when its first 2 coordinates vanish:

$$
\begin{gathered}
\nabla \mathbf{F}=\langle 2 x-2 y+8,-2 x-2 y+4,-1\rangle=\langle 0,0,-1\rangle \Longrightarrow \\
2 x-2 y+8=0 \\
-2 x-2 y+4=0
\end{gathered}
$$

- Adding these equations $\Longrightarrow 4 y=12 \Longrightarrow y=3$.
- Plugging in $y=3$ in first equation gives

$$
2 x-2 \cdot 3+8=0 \Longrightarrow x=-1
$$

- $\mathbf{F}(x, y, z)=x^{2}-2 x y-y^{2}+8 x+4 y-z$ and $\mathbf{F}(-1,3, z)=0$, $\Longrightarrow z=(-1)^{2}-2(-1)(3)-3^{2}+8(-1)+4(3)=2$.
- The unique point with horizontal tangent plane is $(-1,3,2)$.


## Problem 17-Fall 2007

Find the points on the ellipse $x^{2}+4 y^{2}=4$ that are closest to the point $(1,0)$.

## Problem 17-Fall 2007

Find the points on the ellipse $x^{2}+4 y^{2}=4$ that are closest to the point $(1,0)$.

Solution:

- We approach this problem using the method of Lagrange multipliers.


## Problem 17-Fall 2007

Find the points on the ellipse $x^{2}+4 y^{2}=4$ that are closest to the point $(1,0)$.

## Solution:

- We approach this problem using the method of Lagrange multipliers. Let $f(x, y)$ be square of the distance function from $(1,0)$ to an arbitrary point $(x, y)$ in $\mathbb{R}^{2}$.


## Problem 17-Fall 2007

Find the points on the ellipse $x^{2}+4 y^{2}=4$ that are closest to the point $(1,0)$.

## Solution:

- We approach this problem using the method of Lagrange multipliers. Let $f(x, y)$ be square of the distance function from $(1,0)$ to an arbitrary point $(x, y)$ in $\mathbb{R}^{2}$.
- We must find the minimum of $f(x, y)=(x-1)^{2}+y^{2}$, subject to the constraint $g(x, y)=x^{2}+4 y^{2}=4$ (distance squared to $(1,0)$ ).


## Problem 17-Fall 2007

Find the points on the ellipse $x^{2}+4 y^{2}=4$ that are closest to the point $(1,0)$.

## Solution:

- We approach this problem using the method of Lagrange multipliers. Let $f(x, y)$ be square of the distance function from $(1,0)$ to an arbitrary point $(x, y)$ in $\mathbb{R}^{2}$.
- We must find the minimum of $f(x, y)=(x-1)^{2}+y^{2}$, subject to the constraint $g(x, y)=x^{2}+4 y^{2}=4$ (distance squared to $(1,0)$ ).
- Calculating for some $\lambda \in \mathbb{R}$, $\nabla f(x, y)=\langle 2(x-1), 2 y\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle$.


## Problem 17-Fall 2007

Find the points on the ellipse $x^{2}+4 y^{2}=4$ that are closest to the point $(1,0)$.

## Solution:

- We approach this problem using the method of Lagrange multipliers. Let $f(x, y)$ be square of the distance function from $(1,0)$ to an arbitrary point $(x, y)$ in $\mathbb{R}^{2}$.
- We must find the minimum of $f(x, y)=(x-1)^{2}+y^{2}$, subject to the constraint $g(x, y)=x^{2}+4 y^{2}=4$ (distance squared to $(1,0)$ ).
- Calculating for some $\lambda \in \mathbb{R}$, $\nabla f(x, y)=\langle 2(x-1), 2 y\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle$.
- Hence, $2 y=\lambda 8 y$


## Problem 17-Fall 2007

Find the points on the ellipse $x^{2}+4 y^{2}=4$ that are closest to the point $(1,0)$.

## Solution:

- We approach this problem using the method of Lagrange multipliers. Let $f(x, y)$ be square of the distance function from $(1,0)$ to an arbitrary point $(x, y)$ in $\mathbb{R}^{2}$.
- We must find the minimum of $f(x, y)=(x-1)^{2}+y^{2}$, subject to the constraint $g(x, y)=x^{2}+4 y^{2}=4$ (distance squared to $(1,0)$ ).
- Calculating for some $\lambda \in \mathbb{R}$, $\nabla f(x, y)=\langle 2(x-1), 2 y\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle$.
- Hence, $2 y=\lambda 8 y \Longrightarrow \lambda=\frac{1}{4}$ or $y=0$.


## Problem 17-Fall 2007

Find the points on the ellipse $x^{2}+4 y^{2}=4$ that are closest to the point $(1,0)$.

## Solution:

- We approach this problem using the method of Lagrange multipliers. Let $f(x, y)$ be square of the distance function from $(1,0)$ to an arbitrary point $(x, y)$ in $\mathbb{R}^{2}$.
- We must find the minimum of $f(x, y)=(x-1)^{2}+y^{2}$, subject to the constraint $g(x, y)=x^{2}+4 y^{2}=4$ (distance squared to $(1,0)$ ).
- Calculating for some $\lambda \in \mathbb{R}$, $\nabla f(x, y)=\langle 2(x-1), 2 y\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle$.
- Hence, $2 y=\lambda 8 y \Longrightarrow \lambda=\frac{1}{4}$ or $y=0$.
- If $y=0$, then the constraint implies $x= \pm 2$.


## Problem 17-Fall 2007

Find the points on the ellipse $x^{2}+4 y^{2}=4$ that are closest to the point $(1,0)$.

## Solution:

- We approach this problem using the method of Lagrange multipliers. Let $f(x, y)$ be square of the distance function from $(1,0)$ to an arbitrary point $(x, y)$ in $\mathbb{R}^{2}$.
- We must find the minimum of $f(x, y)=(x-1)^{2}+y^{2}$, subject to the constraint $g(x, y)=x^{2}+4 y^{2}=4$ (distance squared to $(1,0)$ ).
- Calculating for some $\lambda \in \mathbb{R}$, $\nabla f(x, y)=\langle 2(x-1), 2 y\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle$.
- Hence, $2 y=\lambda 8 y \Longrightarrow \lambda=\frac{1}{4}$ or $y=0$.
- If $y=0$, then the constraint implies $x= \pm 2$.
- If $\lambda=\frac{1}{4}$, then $2(x-1)=\lambda 2 x=\frac{1}{2} x \Longrightarrow x=\frac{4}{3}$.


## Problem 17-Fall 2007

Find the points on the ellipse $x^{2}+4 y^{2}=4$ that are closest to the point $(1,0)$.

## Solution:

- We approach this problem using the method of Lagrange multipliers. Let $f(x, y)$ be square of the distance function from $(1,0)$ to an arbitrary point $(x, y)$ in $\mathbb{R}^{2}$.
- We must find the minimum of $f(x, y)=(x-1)^{2}+y^{2}$, subject to the constraint $g(x, y)=x^{2}+4 y^{2}=4$ (distance squared to $(1,0)$ ).
- Calculating for some $\lambda \in \mathbb{R}$, $\nabla f(x, y)=\langle 2(x-1), 2 y\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle$.
- Hence, $2 y=\lambda 8 y \Longrightarrow \lambda=\frac{1}{4}$ or $y=0$.
- If $y=0$, then the constraint implies $x= \pm 2$.
- If $\lambda=\frac{1}{4}$, then $2(x-1)=\lambda 2 x=\frac{1}{2} x \Longrightarrow x=\frac{4}{3}$.
- If $x=\frac{4}{3}$, then the constraint implies $y= \pm \frac{\sqrt{5}}{3}$.


## Problem 17 - Fall 2007

Find the points on the ellipse $x^{2}+4 y^{2}=4$ that are closest to the point $(1,0)$.

## Solution:

- We approach this problem using the method of Lagrange multipliers. Let $f(x, y)$ be square of the distance function from $(1,0)$ to an arbitrary point $(x, y)$ in $\mathbb{R}^{2}$.
- We must find the minimum of $f(x, y)=(x-1)^{2}+y^{2}$, subject to the constraint $g(x, y)=x^{2}+4 y^{2}=4$ (distance squared to $(1,0)$ ).
- Calculating for some $\lambda \in \mathbb{R}$, $\nabla f(x, y)=\langle 2(x-1), 2 y\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle$.
- Hence, $2 y=\lambda 8 y \Longrightarrow \lambda=\frac{1}{4}$ or $y=0$.
- If $y=0$, then the constraint implies $x= \pm 2$.
- If $\lambda=\frac{1}{4}$, then $2(x-1)=\lambda 2 x=\frac{1}{2} x \Longrightarrow x=\frac{4}{3}$.
- If $x=\frac{4}{3}$, then the constraint implies $y= \pm \frac{\sqrt{5}}{3}$.
- The function $f(x, y)$ has its minimum value at one of the 4 points $( \pm 2,0)$ and $\left(\frac{4}{3}, \pm \frac{\sqrt{5}}{3}\right)$,


## Problem 17-Fall 2007

Find the points on the ellipse $x^{2}+4 y^{2}=4$ that are closest to the point $(1,0)$.

## Solution:

- We approach this problem using the method of Lagrange multipliers. Let $f(x, y)$ be square of the distance function from $(1,0)$ to an arbitrary point $(x, y)$ in $\mathbb{R}^{2}$.
- We must find the minimum of $f(x, y)=(x-1)^{2}+y^{2}$, subject to the constraint $g(x, y)=x^{2}+4 y^{2}=4$ (distance squared to $(1,0)$ ).
- Calculating for some $\lambda \in \mathbb{R}$, $\nabla f(x, y)=\langle 2(x-1), 2 y\rangle=\lambda \nabla g(x, y)=\lambda\langle 2 x, 8 y\rangle$.
- Hence, $2 y=\lambda 8 y \Longrightarrow \lambda=\frac{1}{4}$ or $y=0$.
- If $y=0$, then the constraint implies $x= \pm 2$.
- If $\lambda=\frac{1}{4}$, then $2(x-1)=\lambda 2 x=\frac{1}{2} x \Longrightarrow x=\frac{4}{3}$.
- If $x=\frac{4}{3}$, then the constraint implies $y= \pm \frac{\sqrt{5}}{3}$.
- The function $f(x, y)$ has its minimum value at one of the 4 points $( \pm 2,0)$ and $\left(\frac{4}{3}, \pm \frac{\sqrt{5}}{3}\right)$, and one easily checks its minimum value of $\frac{2}{3}$ occurs at the 2 points $\left(\frac{4}{3}, \pm \frac{\sqrt{5}}{3}\right)$.


## Problem 18(a) - Fall 2006

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 1 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Problem 18(a) - Fall 2006

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 1 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}
$$

## Problem 18(a) - Fall 2006

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 1 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

## Problem 18(a) - Fall 2006

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 1 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

- Note that when $t=1$, then $x=1$ and $y=2$


## Problem 18(a) - Fall 2006

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 1 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

- Note that when $t=1$, then $x=1$ and $y=2$ and that $\frac{d x}{d t}=3 t^{2}$ and

$$
\frac{d y}{d t}=2 t
$$

## Problem 18(a) - Fall 2006

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 1 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

- Note that when $t=1$, then $x=1$ and $y=2$ and that $\frac{d x}{d t}=3 t^{2}$ and

$$
\frac{d y}{d t}=2 t \Longrightarrow \frac{d x}{d t}(1)=3 \text { and } \frac{d y}{d t}(1)=2
$$

## Problem 18(a) - Fall 2006

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 1 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

- Note that when $t=1$, then $x=1$ and $y=2$ and that $\frac{d x}{d t}=3 t^{2}$ and

$$
\frac{d y}{d t}=2 t \Longrightarrow \frac{d x}{d t}(1)=3 \text { and } \frac{d y}{d t}(1)=2
$$

- Plug in the values in the table into the Chain Rule at $t=1$ :


## Problem 18(a) - Fall 2006

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 1 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

- Note that when $t=1$, then $x=1$ and $y=2$ and that $\frac{d x}{d t}=3 t^{2}$ and

$$
\frac{d y}{d t}=2 t \Longrightarrow \frac{d x}{d t}(1)=3 \text { and } \frac{d y}{d t}(1)=2
$$

- Plug in the values in the table into the Chain Rule at $t=1$ :

$$
f^{\prime}(1)=\frac{\partial f}{\partial x}(1,2) \cdot 3+\frac{\partial f}{\partial y}(1,2) \cdot 2
$$

## Problem 18(a) - Fall 2006

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 1 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

- Note that when $t=1$, then $x=1$ and $y=2$ and that $\frac{d x}{d t}=3 t^{2}$ and

$$
\frac{d y}{d t}=2 t \Longrightarrow \frac{d x}{d t}(1)=3 \text { and } \frac{d y}{d t}(1)=2
$$

- Plug in the values in the table into the Chain Rule at $t=1$ :

$$
f^{\prime}(1)=\frac{\partial f}{\partial x}(1,2) \cdot 3+\frac{\partial f}{\partial y}(1,2) \cdot 2=(-1) \cdot 3+1 \cdot 2
$$

## Problem 18(a) - Fall 2006

Let $f(x, y)$ be a differentiable function with the following values of the partial derivatives $f_{x}(x, y)$ and $f_{y}(x, y)$ at certain points $(x, y)$

| $x$ | $y$ | $f_{x}(x, y)$ | $f_{y}(x, y)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | -2 | 4 |
| -1 | 2 | 3 | -1 |
| 1 | 2 | -1 | 1 |

(You are given more values than you will need for this problem.) Suppose that $x$ and $y$ are functions of variable $t: x=t^{3} ; \quad y=t^{2}+1$, so that we may regard $f$ as a function of $t$. Compute the derivative of $f$ with respect to $t$ when $t=1$.

## Solution:

- By the Chain Rule we have:

$$
f^{\prime}(t)=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}=\frac{\partial f}{\partial x} \cdot 3 t^{2}+\frac{\partial f}{\partial y} \cdot 2 t .
$$

- Note that when $t=1$, then $x=1$ and $y=2$ and that $\frac{d x}{d t}=3 t^{2}$ and

$$
\frac{d y}{d t}=2 t \Longrightarrow \frac{d x}{d t}(1)=3 \text { and } \frac{d y}{d t}(1)=2 .
$$

- Plug in the values in the table into the Chain Rule at $t=1$ :

$$
f^{\prime}(1)=\frac{\partial f}{\partial x}(1,2) \cdot 3+\frac{\partial f}{\partial y}(1,2) \cdot 2=(-1) \cdot 3+1 \cdot 2=-1 .
$$

## Problem 18(b) - Fall 2006

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v, y=2 u-v .
$$

## Problem 18(b) - Fall 2006

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v, y=2 u-v .
$$

## Solution:

- When $u=1$ and $v=1$, then $x=1^{2}+1=2$,

$$
y=2 \cdot 1-1=1, \quad \frac{\partial x}{\partial v}=1 \quad \text { and } \quad \frac{\partial y}{\partial v}=-1
$$

## Problem 18(b) - Fall 2006

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v, y=2 u-v .
$$

## Solution:

- When $u=1$ and $v=1$, then $x=1^{2}+1=2$,

$$
y=2 \cdot 1-1=1, \quad \frac{\partial x}{\partial v}=1 \quad \text { and } \quad \frac{\partial y}{\partial v}=-1
$$

- By the Chain Rule we have:

$$
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v}
$$

## Problem 18(b) - Fall 2006

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v, y=2 u-v .
$$

## Solution:

- When $u=1$ and $v=1$, then $x=1^{2}+1=2$,

$$
y=2 \cdot 1-1=1, \quad \frac{\partial x}{\partial v}=1 \quad \text { and } \quad \frac{\partial y}{\partial v}=-1
$$

- By the Chain Rule we have:

$$
\begin{aligned}
& \frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \\
& =\left(3 x^{2} y^{2}+y^{3}\right)(1)+\left(2 x^{3} y+3 y^{2} x\right)(-1)
\end{aligned}
$$

## Problem 18(b) - Fall 2006

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v, y=2 u-v .
$$

## Solution:

- When $u=1$ and $v=1$, then $x=1^{2}+1=2$,

$$
y=2 \cdot 1-1=1, \quad \frac{\partial x}{\partial v}=1 \quad \text { and } \quad \frac{\partial y}{\partial v}=-1
$$

- By the Chain Rule we have:

$$
\begin{gathered}
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \\
=\left(3 x^{2} y^{2}+y^{3}\right)(1)+\left(2 x^{3} y+3 y^{2} x\right)(-1)=3 x^{2} y^{2}+y^{3}-2 x^{3} y-3 y^{2} x .
\end{gathered}
$$

## Problem 18(b) - Fall 2006

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v, y=2 u-v
$$

## Solution:

- When $u=1$ and $v=1$, then $x=1^{2}+1=2$,

$$
y=2 \cdot 1-1=1, \quad \frac{\partial x}{\partial v}=1 \quad \text { and } \quad \frac{\partial y}{\partial v}=-1
$$

- By the Chain Rule we have:

$$
\begin{gathered}
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \\
=\left(3 x^{2} y^{2}+y^{3}\right)(1)+\left(2 x^{3} y+3 y^{2} x\right)(-1)=3 x^{2} y^{2}+y^{3}-2 x^{3} y-3 y^{2} x .
\end{gathered}
$$

- So for $u=1$ and $v=1$, we get:


## Problem 18(b) - Fall 2006

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v, y=2 u-v
$$

## Solution:

- When $u=1$ and $v=1$, then $x=1^{2}+1=2$,

$$
y=2 \cdot 1-1=1, \quad \frac{\partial x}{\partial v}=1 \quad \text { and } \quad \frac{\partial y}{\partial v}=-1
$$

- By the Chain Rule we have:

$$
\begin{gathered}
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \\
=\left(3 x^{2} y^{2}+y^{3}\right)(1)+\left(2 x^{3} y+3 y^{2} x\right)(-1)=3 x^{2} y^{2}+y^{3}-2 x^{3} y-3 y^{2} x .
\end{gathered}
$$

- So for $u=1$ and $v=1$, we get:

$$
\frac{\partial z}{\partial v}(1,1)
$$

## Problem 18(b) - Fall 2006

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v, y=2 u-v
$$

## Solution:

- When $u=1$ and $v=1$, then $x=1^{2}+1=2$,

$$
y=2 \cdot 1-1=1, \quad \frac{\partial x}{\partial v}=1 \quad \text { and } \quad \frac{\partial y}{\partial v}=-1
$$

- By the Chain Rule we have:

$$
\begin{gathered}
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \\
=\left(3 x^{2} y^{2}+y^{3}\right)(1)+\left(2 x^{3} y+3 y^{2} x\right)(-1)=3 x^{2} y^{2}+y^{3}-2 x^{3} y-3 y^{2} x .
\end{gathered}
$$

- So for $u=1$ and $v=1$, we get:

$$
\frac{\partial z}{\partial v}(1,1)=3 \cdot 4+1-2 \cdot 8-3 \cdot 2
$$

## Problem 18(b) - Fall 2006

Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u=1$ and $v=1$, where

$$
z=x^{3} y^{2}+y^{3} x ; \quad x=u^{2}+v, y=2 u-v
$$

## Solution:

- When $u=1$ and $v=1$, then $x=1^{2}+1=2$,

$$
y=2 \cdot 1-1=1, \quad \frac{\partial x}{\partial v}=1 \quad \text { and } \quad \frac{\partial y}{\partial v}=-1
$$

- By the Chain Rule we have:

$$
\begin{gathered}
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \\
=\left(3 x^{2} y^{2}+y^{3}\right)(1)+\left(2 x^{3} y+3 y^{2} x\right)(-1)=3 x^{2} y^{2}+y^{3}-2 x^{3} y-3 y^{2} x .
\end{gathered}
$$

- So for $u=1$ and $v=1$, we get:

$$
\frac{\partial z}{\partial v}(1,1)=3 \cdot 4+1-2 \cdot 8-3 \cdot 2=-9
$$

## Problem 19(a) - Fall 2006

Let $f(x, y)=x^{2} y^{3}+y^{4}$. Find the directional derivative of $f$ at the point $(1,1)$ in the direction which forms an angle (counterclockwise) of $\pi / 6$ with the positive $x$-axis.

## Problem 19(a) - Fall 2006

Let $f(x, y)=x^{2} y^{3}+y^{4}$. Find the directional derivative of $f$ at the point $(1,1)$ in the direction which forms an angle (counterclockwise) of $\pi / 6$ with the positive $x$-axis.

## Solution:

- The unit vector in the direction of $\frac{\pi}{6}$ is

$$
\mathbf{u}=\left\langle\cos \frac{\pi}{6}, \sin \frac{\pi}{6}\right\rangle=\left\langle\frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle .
$$

## Problem 19(a) - Fall 2006

Let $f(x, y)=x^{2} y^{3}+y^{4}$. Find the directional derivative of $f$ at the point $(1,1)$ in the direction which forms an angle (counterclockwise) of $\pi / 6$ with the positive $x$-axis.

## Solution:

- The unit vector in the direction of $\frac{\pi}{6}$ is

$$
\mathbf{u}=\left\langle\cos \frac{\pi}{6}, \sin \frac{\pi}{6}\right\rangle=\left\langle\frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle .
$$

- Calculating the gradient, we get:

$$
\nabla f(x, y)=\left\langle 2 x y^{3}, 3 x^{2} y^{2}+4 y^{3}\right\rangle
$$

## Problem 19(a) - Fall 2006

Let $f(x, y)=x^{2} y^{3}+y^{4}$. Find the directional derivative of $f$ at the point $(1,1)$ in the direction which forms an angle (counterclockwise) of $\pi / 6$ with the positive $x$-axis.

## Solution:

- The unit vector in the direction of $\frac{\pi}{6}$ is

$$
\mathbf{u}=\left\langle\cos \frac{\pi}{6}, \sin \frac{\pi}{6}\right\rangle=\left\langle\frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle
$$

- Calculating the gradient, we get:

$$
\begin{gathered}
\nabla f(x, y)=\left\langle 2 x y^{3}, 3 x^{2} y^{2}+4 y^{3}\right\rangle \\
\nabla f(1,1)=\langle 2,3+4\rangle=\langle 2,7\rangle
\end{gathered}
$$

## Problem 19(a) - Fall 2006

Let $f(x, y)=x^{2} y^{3}+y^{4}$. Find the directional derivative of $f$ at the point $(1,1)$ in the direction which forms an angle (counterclockwise) of $\pi / 6$ with the positive $x$-axis.

## Solution:

- The unit vector in the direction of $\frac{\pi}{6}$ is

$$
\mathbf{u}=\left\langle\cos \frac{\pi}{6}, \sin \frac{\pi}{6}\right\rangle=\left\langle\frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle
$$

- Calculating the gradient, we get:

$$
\begin{gathered}
\nabla f(x, y)=\left\langle 2 x y^{3}, 3 x^{2} y^{2}+4 y^{3}\right\rangle \\
\nabla f(1,1)=\langle 2,3+4\rangle=\langle 2,7\rangle
\end{gathered}
$$

- So the directional derivative is:

$$
D_{\mathbf{u}} f(1,1)=\nabla f(1,1) \cdot \mathbf{u}
$$

## Problem 19(a) - Fall 2006

Let $f(x, y)=x^{2} y^{3}+y^{4}$. Find the directional derivative of $f$ at the point $(1,1)$ in the direction which forms an angle (counterclockwise) of $\pi / 6$ with the positive $x$-axis.

## Solution:

- The unit vector in the direction of $\frac{\pi}{6}$ is

$$
\mathbf{u}=\left\langle\cos \frac{\pi}{6}, \sin \frac{\pi}{6}\right\rangle=\left\langle\frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle
$$

- Calculating the gradient, we get:

$$
\begin{gathered}
\nabla f(x, y)=\left\langle 2 x y^{3}, 3 x^{2} y^{2}+4 y^{3}\right\rangle \\
\nabla f(1,1)=\langle 2,3+4\rangle=\langle 2,7\rangle
\end{gathered}
$$

- So the directional derivative is:

$$
D_{\mathbf{u}} f(1,1)=\nabla f(1,1) \cdot \mathbf{u}=\langle 2,7\rangle \cdot\left\langle\frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle
$$

## Problem 19(a) - Fall 2006

Let $f(x, y)=x^{2} y^{3}+y^{4}$. Find the directional derivative of $f$ at the point $(1,1)$ in the direction which forms an angle (counterclockwise) of $\pi / 6$ with the positive $x$-axis.

## Solution:

- The unit vector in the direction of $\frac{\pi}{6}$ is

$$
\mathbf{u}=\left\langle\cos \frac{\pi}{6}, \sin \frac{\pi}{6}\right\rangle=\left\langle\frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle
$$

- Calculating the gradient, we get:

$$
\begin{gathered}
\nabla f(x, y)=\left\langle 2 x y^{3}, 3 x^{2} y^{2}+4 y^{3}\right\rangle \\
\nabla f(1,1)=\langle 2,3+4\rangle=\langle 2,7\rangle
\end{gathered}
$$

- So the directional derivative is:

$$
D_{\mathrm{u}} f(1,1)=\nabla f(1,1) \cdot \mathbf{u}=\langle 2,7\rangle \cdot\left\langle\frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle=\sqrt{3}+\frac{7}{2}
$$

## Problem 19(b) - Fall 2006

Find an equation of the tangent line to curve
$x^{2} y+y^{3}-5=0$ at the point $(x, y)=(2,1)$.

## Problem 19(b) - Fall 2006

Find an equation of the tangent line to the curve $x^{2} y+y^{3}-5=0$ at the point $(x, y)=(2,1)$.

## Solution:

- The normal vector $\mathbf{n}$ to the curve $\mathbf{F}(x, y)=x^{2} y+y^{3}-5=0$ at the point $(2,1)$ is $\nabla \mathbf{F}(2,1)$.


## Problem 19(b) - Fall 2006

Find an equation of the tangent line to the curve $x^{2} y+y^{3}-5=0$ at the point $(x, y)=(2,1)$.

## Solution:

- The normal vector $\mathbf{n}$ to the curve $\mathbf{F}(x, y)=x^{2} y+y^{3}-5=0$ at the point $(2,1)$ is $\nabla \mathbf{F}(2,1)$.
- Calculating, we obtain:

$$
\nabla \mathbf{F}(x, y)=\left\langle 2 x y, x^{2}+3 y^{2}\right\rangle
$$

## Problem 19(b) - Fall 2006

Find an equation of the tangent line to the curve $x^{2} y+y^{3}-5=0$ at the point $(x, y)=(2,1)$.

## Solution:

- The normal vector $\mathbf{n}$ to the curve $\mathbf{F}(x, y)=x^{2} y+y^{3}-5=0$ at the point $(2,1)$ is $\nabla \mathbf{F}(2,1)$.
- Calculating, we obtain:

$$
\begin{gathered}
\nabla \mathbf{F}(x, y)=\left\langle 2 x y, x^{2}+3 y^{2}\right\rangle \\
\mathbf{n}=\nabla \mathbf{F}(2,1)=\langle 4,7\rangle
\end{gathered}
$$

## Problem 19(b) - Fall 2006

Find an equation of the tangent line to the curve $x^{2} y+y^{3}-5=0$ at the point $(x, y)=(2,1)$.

## Solution:

- The normal vector $\mathbf{n}$ to the curve $\mathbf{F}(x, y)=x^{2} y+y^{3}-5=0$ at the point $(2,1)$ is $\nabla \mathbf{F}(2,1)$.
- Calculating, we obtain:

$$
\begin{gathered}
\nabla \mathbf{F}(x, y)=\left\langle 2 x y, x^{2}+3 y^{2}\right\rangle \\
\mathbf{n}=\nabla \mathbf{F}(2,1)=\langle 4,7\rangle
\end{gathered}
$$

- The equation of the tangent line is:

$$
\mathbf{n} \cdot\langle x-2, y-1\rangle
$$

## Problem 19(b) - Fall 2006

Find an equation of the tangent line to the curve $x^{2} y+y^{3}-5=0$ at the point $(x, y)=(2,1)$.

## Solution:

- The normal vector $\mathbf{n}$ to the curve $\mathbf{F}(x, y)=x^{2} y+y^{3}-5=0$ at the point $(2,1)$ is $\nabla \mathbf{F}(2,1)$.
- Calculating, we obtain:

$$
\begin{gathered}
\nabla \mathbf{F}(x, y)=\left\langle 2 x y, x^{2}+3 y^{2}\right\rangle \\
\mathbf{n}=\nabla \mathbf{F}(2,1)=\langle 4,7\rangle
\end{gathered}
$$

- The equation of the tangent line is:

$$
\mathbf{n} \cdot\langle x-2, y-1\rangle=\langle 4,7\rangle \cdot\langle x-2, y-1\rangle
$$

## Problem 19(b) - Fall 2006

Find an equation of the tangent line to the curve $x^{2} y+y^{3}-5=0$ at the point $(x, y)=(2,1)$.

## Solution:

- The normal vector $\mathbf{n}$ to the curve $\mathbf{F}(x, y)=x^{2} y+y^{3}-5=0$ at the point $(2,1)$ is $\nabla \mathbf{F}(2,1)$.
- Calculating, we obtain:

$$
\begin{gathered}
\nabla \mathbf{F}(x, y)=\left\langle 2 x y, x^{2}+3 y^{2}\right\rangle \\
\mathbf{n}=\nabla \mathbf{F}(2,1)=\langle 4,7\rangle
\end{gathered}
$$

- The equation of the tangent line is:

$$
\mathbf{n} \cdot\langle x-2, y-1\rangle=\langle 4,7\rangle \cdot\langle x-2, y-1\rangle=4(x-2)+7(y-1)=0 .
$$

## Problem 20 - Fall 2006

Let $\quad f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Problem 20 - Fall 2006

Let

$$
f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}
$$

Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+10 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

## Problem 20 - Fall 2006

Let

$$
f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}
$$

Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+10 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

- This gives the following two equations:


## Problem 20 - Fall 2006

Let

$$
f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}
$$

Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+10 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

- This gives the following two equations:

$$
\begin{aligned}
& 6 x^{2}+y^{2}+10 x=0 \\
& 2 x y+2 y=y(2 x+2)=0
\end{aligned}
$$

## Problem 20 - Fall 2006

Let

$$
f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}
$$

Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+10 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

- This gives the following two equations:

$$
\begin{gathered}
6 x^{2}+y^{2}+10 x=0 \\
2 x y+2 y=y(2 x+2)=0 \Longrightarrow y=0 \text { or } x=-1
\end{gathered}
$$

## Problem 20 - Fall 2006

Let

$$
f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}
$$

Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+10 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

- This gives the following two equations:

$$
\begin{gathered}
6 x^{2}+y^{2}+10 x=0 \\
2 x y+2 y=y(2 x+2)=0 \Longrightarrow y=0 \text { or } x=-1
\end{gathered}
$$

- If $x=-1$, then the first equation gives:

$$
6+y^{2}-10=y^{2}-4=0
$$

## Problem 20 - Fall 2006

Let

$$
f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}
$$

Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+10 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

- This gives the following two equations:

$$
\begin{gathered}
6 x^{2}+y^{2}+10 x=0 \\
2 x y+2 y=y(2 x+2)=0 \Longrightarrow y=0 \text { or } x=-1
\end{gathered}
$$

- If $x=-1$, then the first equation gives:

$$
6+y^{2}-10=y^{2}-4=0 \Longrightarrow y=2 \text { or } y=-2
$$

## Problem 20 - Fall 2006

Let

$$
f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}
$$

Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+10 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

- This gives the following two equations:

$$
\begin{gathered}
6 x^{2}+y^{2}+10 x=0 \\
2 x y+2 y=y(2 x+2)=0 \Longrightarrow y=0 \text { or } x=-1
\end{gathered}
$$

- If $x=-1$, then the first equation gives:

$$
6+y^{2}-10=y^{2}-4=0 \Longrightarrow y=2 \text { or } y=-2
$$

- If $y=0$, then the first equation gives $x=0$ or $x=-\frac{5}{3}$.


## Problem 20 - Fall 2006

Let $\quad f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

## Solution:

- First calculate $\nabla f(x, y)$ and set to $\langle 0,0\rangle$ :

$$
\nabla f(x, y)=\left\langle 6 x^{2}+y^{2}+10 x, 2 x y+2 y\right\rangle=\langle 0,0\rangle .
$$

- This gives the following two equations:

$$
\begin{gathered}
6 x^{2}+y^{2}+10 x=0 \\
2 x y+2 y=y(2 x+2)=0 \Longrightarrow y=0 \text { or } x=-1
\end{gathered}
$$

- If $x=-1$, then the first equation gives:

$$
6+y^{2}-10=y^{2}-4=0 \Longrightarrow y=2 \text { or } y=-2
$$

- If $y=0$, then the first equation gives $x=0$ or $x=-\frac{5}{3}$.
- The set of critical points is:

$$
\left\{(0,0),\left(-\frac{5}{3}, 0\right),(-1,2),(-1,-2)\right\}
$$

Problem 20 - Fall 2006
Let $\quad f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

Solution: Continuation of problem 14.

- Recall that $\left\{(0,0),\left(-\frac{5}{3}, 0\right),(-1,2),(-1,-2)\right\}$ is the set of critical points.

Problem 20 - Fall 2006
Let $\quad f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

Solution: Continuation of problem 14.

- Recall that $\left\{(0,0),\left(-\frac{5}{3}, 0\right),(-1,2),(-1,-2)\right\}$ is the set of critical points.
- Since we will apply the Second Derivative Test, we first write down the second derivative matrix:

$$
\mathbf{D}=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x+10 & 2 y \\
2 y & 2 x+2
\end{array}\right|
$$

Problem 20 - Fall 2006
Let $\quad f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

Solution: Continuation of problem 14.

- Recall that $\left\{(0,0),\left(-\frac{5}{3}, 0\right),(-1,2),(-1,-2)\right\}$ is the set of critical points.
- Since we will apply the Second Derivative Test, we first write down the second derivative matrix:

$$
\mathbf{D}=\left|\begin{array}{cc}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x+10 & 2 y \\
2 y & 2 x+2
\end{array}\right|
$$

- Since $D(0,0)=10 \cdot 2=20>0$ and $f_{x x}(0,0)=10>0$,

Problem 20 - Fall 2006
Let $\quad f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

Solution: Continuation of problem 14.

- Recall that $\left\{(0,0),\left(-\frac{5}{3}, 0\right),(-1,2),(-1,-2)\right\}$ is the set of critical points.
- Since we will apply the Second Derivative Test, we first write down the second derivative matrix:

$$
\mathbf{D}=\left|\begin{array}{cc}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x+10 & 2 y \\
2 y & 2 x+2
\end{array}\right|
$$

- Since $D(0,0)=10 \cdot 2=20>0$ and $f_{x x}(0,0)=10>0$, then $(0,0)$ is a local minimum.


## Problem 20 - Fall 2006

Let $\quad f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

Solution: Continuation of problem 14.

- Recall that $\left\{(0,0),\left(-\frac{5}{3}, 0\right),(-1,2),(-1,-2)\right\}$ is the set of critical points.
- Since we will apply the Second Derivative Test, we first write down the second derivative matrix:

$$
\mathbf{D}=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
12 x+10 & 2 y \\
2 y & 2 x+2
\end{array}\right|
$$

- Since $D(0,0)=10 \cdot 2=20>0$ and $f_{x x}(0,0)=10>0$, then $(0,0)$ is a local minimum.
- Since $D\left(\frac{-5}{3}, 0\right)=-10 \cdot\left(-\frac{4}{3}\right)>0$ and $f_{x x}\left(\frac{-5}{3}, 0\right)=-10<0$,


## Problem 20 - Fall 2006

Let $\quad f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

Solution: Continuation of problem 14.

- Recall that $\left\{(0,0),\left(-\frac{5}{3}, 0\right),(-1,2),(-1,-2)\right\}$ is the set of critical points.
- Since we will apply the Second Derivative Test, we first write down the second derivative matrix:

$$
\mathbf{D}=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left\lvert\, \begin{array}{cc}
12 x+10 & 2 y \\
2 y & 2 x+2
\end{array}\right.
$$

- Since $D(0,0)=10 \cdot 2=20>0$ and $f_{x x}(0,0)=10>0$, then $(0,0)$ is a local minimum.
- Since $D\left(\frac{-5}{3}, 0\right)=-10 \cdot\left(-\frac{4}{3}\right)>0$ and $f_{x x}\left(\frac{-5}{3}, 0\right)=-10<0$, then $\left(-\frac{5}{3}, 0\right)$ is a local maximum.


## Problem 20 - Fall 2006

Let $\quad f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

Solution: Continuation of problem 14.

- Recall that $\left\{(0,0),\left(-\frac{5}{3}, 0\right),(-1,2),(-1,-2)\right\}$ is the set of critical points.
- Since we will apply the Second Derivative Test, we first write down the second derivative matrix:

$$
\mathbf{D}=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left\lvert\, \begin{array}{cc}
12 x+10 & 2 y \\
2 y & 2 x+2
\end{array}\right.
$$

- Since $D(0,0)=10 \cdot 2=20>0$ and $f_{x x}(0,0)=10>0$, then $(0,0)$ is a local minimum.
- Since $D\left(\frac{-5}{3}, 0\right)=-10 \cdot\left(-\frac{4}{3}\right)>0$ and $f_{x x}\left(\frac{-5}{3}, 0\right)=-10<0$, then $\left(-\frac{5}{3}, 0\right)$ is a local maximum.
- Since $D(-1,2)<0$, then $(-1,2)$ is a saddle point.


## Problem 20 - Fall 2006

Let $\quad f(x, y)=2 x^{3}+x y^{2}+5 x^{2}+y^{2}$.
Find and classify (as local maxima, local minima or saddle points) all critical points of $f$.

Solution: Continuation of problem 14.

- Recall that $\left\{(0,0),\left(-\frac{5}{3}, 0\right),(-1,2),(-1,-2)\right\}$ is the set of critical points.
- Since we will apply the Second Derivative Test, we first write down the second derivative matrix:

$$
\mathbf{D}=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left\lvert\, \begin{array}{cc}
12 x+10 & 2 y \\
2 y & 2 x+2
\end{array}\right.
$$

- Since $D(0,0)=10 \cdot 2=20>0$ and $f_{x x}(0,0)=10>0$, then $(0,0)$ is a local minimum.
- Since $D\left(\frac{-5}{3}, 0\right)=-10 \cdot\left(-\frac{4}{3}\right)>0$ and $f_{x x}\left(\frac{-5}{3}, 0\right)=-10<0$, then $\left(-\frac{5}{3}, 0\right)$ is a local maximum.
- Since $D(-1,2)<0$, then $(-1,2)$ is a saddle point.
- Since $D(-1,-2)<0$, then $(-1,-2)$ is a saddle point.


## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Solution:

- The constraint function is $g(x, y)=x^{2}+y^{2}$.


## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Solution:

- The constraint function is $g(x, y)=x^{2}+y^{2}$. Note that $x$ and $y$ cannot both be 0 .


## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Solution:

- The constraint function is $g(x, y)=x^{2}+y^{2}$. Note that $x$ and $y$ cannot both be 0 .
- Set $\nabla f=\langle 4 x, 2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$ and solve:


## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Solution:

- The constraint function is $g(x, y)=x^{2}+y^{2}$. Note that $x$ and $y$ cannot both be 0 .
- Set $\nabla f=\langle 4 x, 2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$ and solve:

$$
4 x=\lambda 2 x
$$

## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Solution:

- The constraint function is $g(x, y)=x^{2}+y^{2}$. Note that $x$ and $y$ cannot both be 0 .
- Set $\nabla f=\langle 4 x, 2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$ and solve:

$$
4 x=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=2
$$

## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Solution:

- The constraint function is $g(x, y)=x^{2}+y^{2}$. Note that $x$ and $y$ cannot both be 0 .
- Set $\nabla f=\langle 4 x, 2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$ and solve:

$$
\begin{aligned}
& 4 x=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=2 \\
& 2 y=\lambda 2 y
\end{aligned}
$$

## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Solution:

- The constraint function is $g(x, y)=x^{2}+y^{2}$. Note that $x$ and $y$ cannot both be 0 .
- Set $\nabla f=\langle 4 x, 2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$ and solve:

$$
\begin{aligned}
& 4 x=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=2 \\
& 2 y=\lambda 2 y \Longrightarrow y=0 \text { or } \lambda=1
\end{aligned}
$$

## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Solution:

- The constraint function is $g(x, y)=x^{2}+y^{2}$. Note that $x$ and $y$ cannot both be 0 .
- Set $\nabla f=\langle 4 x, 2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$ and solve:

$$
\begin{aligned}
& 4 x=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=2 \\
& 2 y=\lambda 2 y \Longrightarrow y=0 \text { or } \lambda=1
\end{aligned}
$$

- Since $\lambda$ cannot simultaneously be 2 and 1 , then $x$ or $y$ is zero.


## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Solution:

- The constraint function is $g(x, y)=x^{2}+y^{2}$. Note that $x$ and $y$ cannot both be 0 .
- Set $\nabla f=\langle 4 x, 2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$ and solve:

$$
\begin{aligned}
& 4 x=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=2 \\
& 2 y=\lambda 2 y \Longrightarrow y=0 \text { or } \lambda=1
\end{aligned}
$$

- Since $\lambda$ cannot simultaneously be 2 and 1 , then $x$ or $y$ is zero.
- From the constraint $x^{2}+y^{2}=1$,


## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Solution:

- The constraint function is $g(x, y)=x^{2}+y^{2}$. Note that $x$ and $y$ cannot both be 0 .
- Set $\nabla f=\langle 4 x, 2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$ and solve:

$$
\begin{aligned}
& 4 x=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=2 \\
& 2 y=\lambda 2 y \Longrightarrow y=0 \text { or } \lambda=1
\end{aligned}
$$

- Since $\lambda$ cannot simultaneously be 2 and 1 , then $x$ or $y$ is zero.
- From the constraint $x^{2}+y^{2}=1, x=0 \Longrightarrow y= \pm 1$


## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Solution:

- The constraint function is $g(x, y)=x^{2}+y^{2}$. Note that $x$ and $y$ cannot both be 0 .
- Set $\nabla f=\langle 4 x, 2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$ and solve:

$$
\begin{aligned}
& 4 x=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=2 \\
& 2 y=\lambda 2 y \Longrightarrow y=0 \text { or } \lambda=1
\end{aligned}
$$

- Since $\lambda$ cannot simultaneously be 2 and 1 , then $x$ or $y$ is zero.
- From the constraint $x^{2}+y^{2}=1, x=0 \Longrightarrow y= \pm 1$ and $y=0 \Longrightarrow x= \pm 1$.


## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Solution:

- The constraint function is $g(x, y)=x^{2}+y^{2}$. Note that $x$ and $y$ cannot both be 0 .
- Set $\nabla f=\langle 4 x, 2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$ and solve:

$$
\begin{aligned}
& 4 x=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=2 \\
& 2 y=\lambda 2 y \Longrightarrow y=0 \text { or } \lambda=1
\end{aligned}
$$

- Since $\lambda$ cannot simultaneously be 2 and 1 , then $x$ or $y$ is zero.
- From the constraint $x^{2}+y^{2}=1, x=0 \Longrightarrow y= \pm 1$ and $y=0 \Longrightarrow x= \pm 1$.
- We only need to check the values of $f$ at the points $(0, \pm 1)$, $( \pm 1,0)$ :


## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Solution:

- The constraint function is $g(x, y)=x^{2}+y^{2}$. Note that $x$ and $y$ cannot both be 0 .
- Set $\nabla f=\langle 4 x, 2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$ and solve:

$$
\begin{aligned}
& 4 x=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=2 \\
& 2 y=\lambda 2 y \Longrightarrow y=0 \text { or } \lambda=1
\end{aligned}
$$

- Since $\lambda$ cannot simultaneously be 2 and 1 , then $x$ or $y$ is zero.
- From the constraint $x^{2}+y^{2}=1, x=0 \Longrightarrow y= \pm 1$ and $y=0 \Longrightarrow x= \pm 1$.
- We only need to check the values of $f$ at the points $(0, \pm 1)$, $( \pm 1,0)$ :

$$
f(0, \pm 1)=1 \quad f( \pm 1,0)=2
$$

## Problem 21 - Fall 2006

Find the maximum value of $f(x, y)=2 x^{2}+y^{2}$ on the circle $x^{2}+y^{2}=1$ (Hint: Use Lagrange Multipliers).

## Solution:

- The constraint function is $g(x, y)=x^{2}+y^{2}$. Note that $x$ and $y$ cannot both be 0 .
- Set $\nabla f=\langle 4 x, 2 y\rangle=\lambda \nabla g=\lambda\langle 2 x, 2 y\rangle$ and solve:

$$
\begin{aligned}
& 4 x=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=2 \\
& 2 y=\lambda 2 y \Longrightarrow y=0 \text { or } \lambda=1
\end{aligned}
$$

- Since $\lambda$ cannot simultaneously be 2 and 1 , then $x$ or $y$ is zero.
- From the constraint $x^{2}+y^{2}=1, x=0 \Longrightarrow y= \pm 1$ and $y=0 \Longrightarrow x= \pm 1$.
- We only need to check the values of $f$ at the points $(0, \pm 1)$, $( \pm 1,0)$ :

$$
f(0, \pm 1)=1 \quad f( \pm 1,0)=2
$$

- $f(x, y)$ has its maximum value 2 at the points $( \pm 1,0)$.


## Problem 22 - Fall 2006

Find the volume $\mathbf{V}$ above the rectangle $-1 \leq x \leq 1$, $2 \leq y \leq 5$ and below the surface $z=5+x^{2}+y$.

## Problem 22 - Fall 2006

Find the volume V above the rectangle $-1 \leq x \leq 1$, $2 \leq y \leq 5$ and below the surface $z=5+x^{2}+y$.

## Solution:

We apply Fubini's Theorem:

$$
\mathbf{V}=\int_{2}^{5} \int_{-1}^{1}\left(5+x^{2}+y\right) d x d y
$$

## Problem 22 - Fall 2006

Find the volume $\mathbf{V}$ above the rectangle $-1 \leq x \leq 1$, $2 \leq y \leq 5$ and below the surface $z=5+x^{2}+y$.

## Solution:

We apply Fubini's Theorem:

$$
\mathbf{V}=\int_{2}^{5} \int_{-1}^{1}\left(5+x^{2}+y\right) d x d y=\int_{2}^{5}\left[5 x+\frac{x^{3}}{3}+y x\right]_{-1}^{1} d y
$$

## Problem 22 - Fall 2006

Find the volume $\mathbf{V}$ above the rectangle $-1 \leq x \leq 1$, $2 \leq y \leq 5$ and below the surface $z=5+x^{2}+y$.

## Solution:

We apply Fubini's Theorem:

$$
\begin{aligned}
\mathbf{V}= & \int_{2}^{5} \int_{-1}^{1}\left(5+x^{2}+y\right) d x d y=\int_{2}^{5}\left[5 x+\frac{x^{3}}{3}+y x\right]_{-1}^{1} d y \\
& =\int_{2}^{5} 10+\frac{2}{3}+2 y d y
\end{aligned}
$$

## Problem 22 - Fall 2006

Find the volume V above the rectangle $-1 \leq x \leq 1$, $2 \leq y \leq 5$ and below the surface $z=5+x^{2}+y$.

## Solution:

We apply Fubini's Theorem:

$$
\begin{aligned}
\mathbf{V}= & \int_{2}^{5} \int_{-1}^{1}\left(5+x^{2}+y\right) d x d y=\int_{2}^{5}\left[5 x+\frac{x^{3}}{3}+y x\right]_{-1}^{1} d y \\
& =\int_{2}^{5} 10+\frac{2}{3}+2 y d y=\left(10+\frac{2}{3}\right) y+\left.y^{2}\right|_{2} ^{5}
\end{aligned}
$$

## Problem 22 - Fall 2006

Find the volume V above the rectangle $-1 \leq x \leq 1$, $2 \leq y \leq 5$ and below the surface $z=5+x^{2}+y$.

## Solution:

We apply Fubini's Theorem:

$$
\begin{aligned}
\mathbf{V}= & \int_{2}^{5} \int_{-1}^{1}\left(5+x^{2}+y\right) d x d y=\int_{2}^{5}\left[5 x+\frac{x^{3}}{3}+y x\right]_{-1}^{1} d y \\
& =\int_{2}^{5} 10+\frac{2}{3}+2 y d y=\left(10+\frac{2}{3}\right) y+\left.y^{2}\right|_{2} ^{5}=53
\end{aligned}
$$

## Problem 23 - Fall 2006

Evaluate the integral

$$
\int_{0}^{1} \int_{\sqrt{y}}^{1} \sqrt{x^{3}+1} d x d y
$$

by reversing the order of integration.

## Problem 23 - Fall 2006

Evaluate the integral

$$
\int_{0}^{1} \int_{\sqrt{y}}^{1} \sqrt{x^{3}+1} d x d y
$$

by reversing the order of integration.

## Solution:

There is no integration problem on this exam with varying limits of integration (function limits).

## Problem 24(1)

Use Chain Rule to find $d z / d t$.
$z=x^{2} y+2 y^{3}, x=1+t^{2}, y=(1-t)^{2}$.

## Problem 24(1)

Use Chain Rule to find $d z / d t$.
$z=x^{2} y+2 y^{3}, x=1+t^{2}, y=(1-t)^{2}$.
Solution:

- Calculating:

$$
\begin{array}{ll}
\frac{d x}{d t}=2 t & \frac{d y}{d t}=-2(1-t) \\
\frac{\partial z}{\partial x}=2 x y & \frac{\partial z}{\partial y}=x^{2}+6 y^{2}
\end{array}
$$

## Problem 24(1)

Use Chain Rule to find $d z / d t$.
$z=x^{2} y+2 y^{3}, x=1+t^{2}, y=(1-t)^{2}$.
Solution:

- Calculating:

$$
\begin{array}{ll}
\frac{d x}{d t}=2 t & \frac{d y}{d t}=-2(1-t) \\
\frac{\partial z}{\partial x}=2 x y & \frac{\partial z}{\partial y}=x^{2}+6 y^{2}
\end{array}
$$

- By the Chain Rule,

$$
\frac{d z}{d t}=\frac{\partial z}{\partial x} \frac{d x}{d t}+\frac{\partial z}{\partial y} \frac{d y}{d t}
$$

## Problem 24(1)

Use Chain Rule to find $d z / d t$.
$z=x^{2} y+2 y^{3}, x=1+t^{2}, y=(1-t)^{2}$.

## Solution:

- Calculating:

$$
\begin{array}{ll}
\frac{d x}{d t}=2 t & \frac{d y}{d t}=-2(1-t) \\
\frac{\partial z}{\partial x}=2 x y & \frac{\partial z}{\partial y}=x^{2}+6 y^{2}
\end{array}
$$

- By the Chain Rule,

$$
\frac{d z}{d t}=\frac{\partial z}{\partial x} \frac{d x}{d t}+\frac{\partial z}{\partial y} \frac{d y}{d t}=2 x y \cdot 2 t+\left(x^{2}+6 y^{2}\right) \cdot(-2(1-t))
$$

## Problem 24(1)

Use Chain Rule to find $d z / d t$.
$z=x^{2} y+2 y^{3}, x=1+t^{2}, y=(1-t)^{2}$.

## Solution:

- Calculating:

$$
\begin{array}{ll}
\frac{d x}{d t}=2 t & \frac{d y}{d t}=-2(1-t) \\
\frac{\partial z}{\partial x}=2 x y & \frac{\partial z}{\partial y}=x^{2}+6 y^{2}
\end{array}
$$

- By the Chain Rule,

$$
\begin{aligned}
& \frac{d z}{d t}=\frac{\partial z}{\partial x} \frac{d x}{d t}+\frac{\partial z}{\partial y} \frac{d y}{d t}=2 x y \cdot 2 t+\left(x^{2}+6 y^{2}\right) \cdot(-2(1-t)) \\
& =2\left(1+t^{2}\right)(1-t)^{2} 2 t+\left(\left(1+t^{2}\right)^{2}+6(1-t)^{4}\right)(-2(1-t))
\end{aligned}
$$

## Problem 24(1)

Use Chain Rule to find $d z / d t$.
$z=x^{2} y+2 y^{3}, x=1+t^{2}, y=(1-t)^{2}$.

## Solution:

- Calculating:

$$
\begin{array}{ll}
\frac{d x}{d t}=2 t & \frac{d y}{d t}=-2(1-t) \\
\frac{\partial z}{\partial x}=2 x y & \frac{\partial z}{\partial y}=x^{2}+6 y^{2}
\end{array}
$$

- By the Chain Rule,

$$
\begin{aligned}
& \frac{d z}{d t}=\frac{\partial z}{\partial x} \frac{d x}{d t}+\frac{\partial z}{\partial y} \frac{d y}{d t}=2 x y \cdot 2 t+\left(x^{2}+6 y^{2}\right) \cdot(-2(1-t)) \\
& =2\left(1+t^{2}\right)(1-t)^{2} 2 t+\left(\left(1+t^{2}\right)^{2}+6(1-t)^{4}\right)(-2(1-t))
\end{aligned}
$$

- You can simplify further if you want.

Problem 24(2)
Use Chain Rule to find $\partial z / \partial u$ and $\partial z / \partial v$.
$z=x^{3}+x y^{2}+y^{3}, x=u v, y=u+v$.

Problem 24(2)
Use Chain Rule to find $\partial z / \partial u$ and $\partial z / \partial v$.
$z=x^{3}+x y^{2}+y^{3}, x=u v, y=u+v$.

## Solution:

- Calculating:

$$
\begin{aligned}
\frac{\partial x}{\partial u}=v & \frac{\partial x}{\partial v}=u \\
\frac{\partial y}{\partial u}=1 & \frac{\partial y}{\partial v}=1 \\
\frac{\partial z}{\partial x}=3 x^{2}+y^{2} & \frac{\partial z}{\partial y}=2 x y+3 y^{2}
\end{aligned}
$$

Problem 24(2)
Use Chain Rule to find $\partial z / \partial u$ and $\partial z / \partial v$.
$z=x^{3}+x y^{2}+y^{3}, x=u v, y=u+v$.

## Solution:

- Calculating:

$$
\begin{aligned}
\frac{\partial x}{\partial u}=v & \frac{\partial x}{\partial v}=u \\
\frac{\partial y}{\partial u}=1 & \frac{\partial y}{\partial v}=1 \\
\frac{\partial z}{\partial x}=3 x^{2}+y^{2} & \frac{\partial z}{\partial y}=2 x y+3 y^{2} .
\end{aligned}
$$

- By the Chain Rule:

$$
\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial u}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial u}
$$

Problem 24(2)
Use Chain Rule to find $\partial z / \partial u$ and $\partial z / \partial v$.
$z=x^{3}+x y^{2}+y^{3}, x=u v, y=u+v$.

## Solution:

- Calculating:

$$
\begin{aligned}
\frac{\partial x}{\partial u}=v & \frac{\partial x}{\partial v}=u \\
\frac{\partial y}{\partial u}=1 & \frac{\partial y}{\partial v}=1 \\
\frac{\partial z}{\partial x}=3 x^{2}+y^{2} & \frac{\partial z}{\partial y}=2 x y+3 y^{2} .
\end{aligned}
$$

- By the Chain Rule:

$$
\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial u}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial u}
$$

$$
=\left(3 x^{2}+y^{2}\right) v+\left(2 x y+3 y^{2}\right)
$$

Problem 24(2)
Use Chain Rule to find $\partial z / \partial u$ and $\partial z / \partial v$.
$z=x^{3}+x y^{2}+y^{3}, x=u v, y=u+v$.

## Solution:

- Calculating:

$$
\begin{aligned}
\frac{\partial x}{\partial u}=v & \frac{\partial x}{\partial v}=u \\
\frac{\partial y}{\partial u}=1 & \frac{\partial y}{\partial v}=1 \\
\frac{\partial z}{\partial x}=3 x^{2}+y^{2} & \frac{\partial z}{\partial y}=2 x y+3 y^{2} .
\end{aligned}
$$

- By the Chain Rule:

$$
\begin{gathered}
\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial u}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \\
=\left(3 x^{2}+y^{2}\right) v+\left(2 x y+3 y^{2}\right)=\left(3 u^{2} v^{2}+(u+v)^{2}\right) v+2 u v(u+v)+3(u+v)^{2},
\end{gathered}
$$

Problem 24(2)
Use Chain Rule to find $\partial z / \partial u$ and $\partial z / \partial v$.
$z=x^{3}+x y^{2}+y^{3}, x=u v, y=u+v$.

## Solution:

- Calculating:

$$
\begin{aligned}
\frac{\partial x}{\partial u}=v & \frac{\partial x}{\partial v}=u \\
\frac{\partial y}{\partial u}=1 & \frac{\partial y}{\partial v}=1 \\
\frac{\partial z}{\partial x}=3 x^{2}+y^{2} & \frac{\partial z}{\partial y}=2 x y+3 y^{2} .
\end{aligned}
$$

- By the Chain Rule:

$$
\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial u}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial u}
$$

$$
=\left(3 x^{2}+y^{2}\right) v+\left(2 x y+3 y^{2}\right)=\left(3 u^{2} v^{2}+(u+v)^{2}\right) v+2 u v(u+v)+3(u+v)^{2},
$$

$$
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v}
$$

Problem 24(2)
Use Chain Rule to find $\partial z / \partial u$ and $\partial z / \partial v$.
$z=x^{3}+x y^{2}+y^{3}, x=u v, y=u+v$.

## Solution:

- Calculating:

$$
\begin{aligned}
\frac{\partial x}{\partial u}=v & \frac{\partial x}{\partial v}=u \\
\frac{\partial y}{\partial u}=1 & \frac{\partial y}{\partial v}=1 \\
\frac{\partial z}{\partial x}=3 x^{2}+y^{2} & \frac{\partial z}{\partial y}=2 x y+3 y^{2} .
\end{aligned}
$$

- By the Chain Rule:

$$
\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial u}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial u}
$$

$$
=\left(3 x^{2}+y^{2}\right) v+\left(2 x y+3 y^{2}\right)=\left(3 u^{2} v^{2}+(u+v)^{2}\right) v+2 u v(u+v)+3(u+v)^{2},
$$

$$
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v}
$$

$=\left(3 x^{2}+y^{2}\right) u+\left(2 x y+3 y^{2}\right)$

## Problem 24(2)

Use Chain Rule to find $\partial z / \partial u$ and $\partial z / \partial v$.
$z=x^{3}+x y^{2}+y^{3}, x=u v, y=u+v$.

## Solution:

- Calculating:

$$
\begin{aligned}
\frac{\partial x}{\partial u}=v & \frac{\partial x}{\partial v}=u \\
\frac{\partial y}{\partial u}=1 & \frac{\partial y}{\partial v}=1 \\
\frac{\partial z}{\partial x}=3 x^{2}+y^{2} & \frac{\partial z}{\partial y}=2 x y+3 y^{2} .
\end{aligned}
$$

- By the Chain Rule:

$$
\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial u}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial u}
$$

$$
=\left(3 x^{2}+y^{2}\right) v+\left(2 x y+3 y^{2}\right)=\left(3 u^{2} v^{2}+(u+v)^{2}\right) v+2 u v(u+v)+3(u+v)^{2},
$$

$$
\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v}
$$

$$
=\left(3 x^{2}+y^{2}\right) u+\left(2 x y+3 y^{2}\right)=\left(3 u^{2} v^{2}+(u+v)^{2}\right) u+\left(2 u v(u+v)+3(u+v)^{2}\right) .
$$

Problem 25
If $z=f(x, y)$, where $f$ is differentiable, and $x=1+t^{2}, y=3 t$, compute $d z / d t$ at $t=2$ provided that $f_{x}(5,6)=f_{y}(5,6)=-1$.

## Problem 25

If $z=f(x, y)$, where $f$ is differentiable, and $x=1+t^{2}, y=3 t$, compute $d z / d t$ at $t=2$ provided that $f_{x}(5,6)=f_{y}(5,6)=-1$.

## Solution:

- We apply the Chain Rule $\frac{d z}{d t}=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}$.


## Problem 25

If $z=f(x, y)$, where $f$ is differentiable, and $x=1+t^{2}, y=3 t$, compute $d z / d t$ at $t=2$ provided that $f_{x}(5,6)=f_{y}(5,6)=-1$.

## Solution:

- We apply the Chain Rule $\frac{d z}{d t}=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}$.
- Since $t=2$, then $x(2)=1+2^{2}=5$ and $y(2)=3 \cdot 2=6$.


## Problem 25

If $z=f(x, y)$, where $f$ is differentiable, and $x=1+t^{2}, y=3 t$, compute $d z / d t$ at $t=2$ provided that $f_{x}(5,6)=f_{y}(5,6)=-1$.

## Solution:

- We apply the Chain Rule $\frac{d z}{d t}=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}$.
- Since $t=2$, then $x(2)=1+2^{2}=5$ and $y(2)=3 \cdot 2=6$.
- Calculating, we obtain:

$$
\frac{d x}{d t}=2 t \quad \frac{d y}{d t}=3
$$

## Problem 25

If $z=f(x, y)$, where $f$ is differentiable, and $x=1+t^{2}, y=3 t$, compute $d z / d t$ at $t=2$ provided that $f_{x}(5,6)=f_{y}(5,6)=-1$.

## Solution:

- We apply the Chain Rule $\frac{d z}{d t}=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}$.
- Since $t=2$, then $x(2)=1+2^{2}=5$ and $y(2)=3 \cdot 2=6$.
- Calculating, we obtain:

$$
\frac{d x}{d t}=2 t \quad \frac{d y}{d t}=3
$$

- Evaluate using the Chain Rule:

$$
\frac{d z}{d t}(2)
$$

## Problem 25

If $z=f(x, y)$, where $f$ is differentiable, and $x=1+t^{2}, y=3 t$, compute $d z / d t$ at $t=2$ provided that $f_{x}(5,6)=f_{y}(5,6)=-1$.

## Solution:

- We apply the Chain Rule $\frac{d z}{d t}=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}$.
- Since $t=2$, then $x(2)=1+2^{2}=5$ and $y(2)=3 \cdot 2=6$.
- Calculating, we obtain:

$$
\frac{d x}{d t}=2 t \quad \frac{d y}{d t}=3 .
$$

- Evaluate using the Chain Rule:

$$
\frac{d z}{d t}(2)=\frac{\partial f}{\partial x}(5,6) \frac{d x}{d t}(2)+\frac{\partial f}{\partial y}(5,6) \frac{d y}{d t}(2)
$$

## Problem 25

If $z=f(x, y)$, where $f$ is differentiable, and $x=1+t^{2}, y=3 t$, compute $d z / d t$ at $t=2$ provided that $f_{x}(5,6)=f_{y}(5,6)=-1$.

## Solution:

- We apply the Chain Rule $\frac{d z}{d t}=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}$.
- Since $t=2$, then $x(2)=1+2^{2}=5$ and $y(2)=3 \cdot 2=6$.
- Calculating, we obtain:

$$
\frac{d x}{d t}=2 t \quad \frac{d y}{d t}=3
$$

- Evaluate using the Chain Rule:

$$
\begin{aligned}
\frac{d z}{d t}(2) & =\frac{\partial f}{\partial x}(5,6) \frac{d x}{d t}(2)+\frac{\partial f}{\partial y}(5,6) \frac{d y}{d t}(2) \\
& =-1(2 \cdot 2)+(-1) 3
\end{aligned}
$$

## Problem 25

If $z=f(x, y)$, where $f$ is differentiable, and $x=1+t^{2}, y=3 t$, compute $d z / d t$ at $t=2$ provided that $f_{x}(5,6)=f_{y}(5,6)=-1$.

## Solution:

- We apply the Chain Rule $\frac{d z}{d t}=\frac{\partial f}{\partial x} \frac{d x}{d t}+\frac{\partial f}{\partial y} \frac{d y}{d t}$.
- Since $t=2$, then $x(2)=1+2^{2}=5$ and $y(2)=3 \cdot 2=6$.
- Calculating, we obtain:

$$
\frac{d x}{d t}=2 t \quad \frac{d y}{d t}=3 .
$$

- Evaluate using the Chain Rule:

$$
\begin{aligned}
\frac{d z}{d t}(2) & =\frac{\partial f}{\partial x}(5,6) \frac{d x}{d t}(2)+\frac{\partial f}{\partial y}(5,6) \frac{d y}{d t}(2) \\
& =-1(2 \cdot 2)+(-1) 3=-7 .
\end{aligned}
$$

## Problem 26(a)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the gradient at $(0,1)$.

## Problem 26(a)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the gradient at $(0,1)$.

## Solution:

(1) $\nabla f(x, y)=\left\langle 2 x y, x^{2}+3 y^{2}-2 y\right\rangle$
$\nabla f(0,1)=\langle 0,1\rangle ;$

## Problem 26(a)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the gradient at $(0,1)$.

## Solution:

$$
\text { (1) } \begin{aligned}
& \nabla f(x, y)=\left\langle 2 x y, x^{2}+3 y^{2}-2 y\right\rangle \\
& \nabla f(0,1)=\langle 0,1\rangle
\end{aligned}
$$

(2) $\nabla g(x, y)=\left\langle\frac{1}{y}+y,-\frac{x}{y^{2}}+x\right\rangle$
$\nabla g(0,1)=\langle 2,0\rangle ;$

## Problem 26(a)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the gradient at $(0,1)$.

Solution:

$$
\text { (1) } \begin{aligned}
& \nabla f(x, y)=\left\langle 2 x y, x^{2}+3 y^{2}-2 y\right\rangle \\
& \nabla f(0,1)=\langle 0,1\rangle ; \\
& \text { (2) } \nabla g(x, y)=\left\langle\frac{1}{y}+y,-\frac{x}{y^{2}}+x\right\rangle \\
& \nabla g(0,1)=\langle 2,0\rangle ; \\
& \text { () } \nabla h(x, y)=\left\langle\cos \left(x^{2} y\right)(2 x y)+y^{2}, \cos \left(x^{2} y\right) x^{2}+2 x y\right\rangle \\
& \nabla h(0,1)=\langle 1,0\rangle .
\end{aligned}
$$

## Problem 26(b)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v}=\langle 3,4\rangle$.

## Problem 26(b)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v}=\langle 3,4\rangle$.

## Solution:

(1) The unit vector $\mathbf{u}$ in the direction of $\mathbf{v}=\langle 3,4\rangle$ is:

$$
\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}
$$

## Problem 26(b)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v}=\langle 3,4\rangle$.

## Solution:

(1) The unit vector $\mathbf{u}$ in the direction of $\mathbf{v}=\langle 3,4\rangle$ is:

$$
\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle
$$

## Problem 26(b)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v}=\langle 3,4\rangle$.

## Solution:

(1) The unit vector $\mathbf{u}$ in the direction of $\mathbf{v}=\langle 3,4\rangle$ is:

$$
\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle=\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle .
$$

## Problem 26(b)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v}=\langle 3,4\rangle$.

## Solution:

(1) The unit vector $\mathbf{u}$ in the direction of $\mathbf{v}=\langle 3,4\rangle$ is:

$$
\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle=\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle .
$$

(2) $D_{\mathrm{u}} f(0,1)=\nabla f(0,1) \cdot \mathbf{u}$

## Problem 26(b)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v}=\langle 3,4\rangle$.

## Solution:

(1) The unit vector $\mathbf{u}$ in the direction of $\mathbf{v}=\langle 3,4\rangle$ is:

$$
\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle=\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle .
$$

(2) $D_{\mathrm{u}} f(0,1)=\nabla f(0,1) \cdot \mathbf{u}=\langle 0,1\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle$

## Problem 26(b)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v}=\langle 3,4\rangle$.

## Solution:

(1) The unit vector $\mathbf{u}$ in the direction of $\mathbf{v}=\langle 3,4\rangle$ is:

$$
\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle=\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle .
$$

(2) $D_{\mathrm{u}} f(0,1)=\nabla f(0,1) \cdot \mathbf{u}=\langle 0,1\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle=\frac{4}{5}$.

## Problem 26(b)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v}=\langle 3,4\rangle$.

## Solution:

(1) The unit vector $\mathbf{u}$ in the direction of $\mathbf{v}=\langle 3,4\rangle$ is:

$$
\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle=\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle .
$$

(2) $D_{\mathrm{u}} f(0,1)=\nabla f(0,1) \cdot \mathbf{u}=\langle 0,1\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle=\frac{4}{5}$.
(3) $D_{\mathrm{u}} g(0,1)=\nabla g(0,1) \cdot \mathbf{u}$

## Problem 26(b)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v}=\langle 3,4\rangle$.

## Solution:

(1) The unit vector $\mathbf{u}$ in the direction of $\mathbf{v}=\langle 3,4\rangle$ is:

$$
\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle=\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle .
$$

(2) $D_{\mathrm{u}} f(0,1)=\nabla f(0,1) \cdot \mathbf{u}=\langle 0,1\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle=\frac{4}{5}$.
(3) $D_{u} g(0,1)=\nabla g(0,1) \cdot \mathbf{u}=\langle 2,0\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle$

## Problem 26(b)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v}=\langle 3,4\rangle$.

## Solution:

(1) The unit vector $\mathbf{u}$ in the direction of $\mathbf{v}=\langle 3,4\rangle$ is:

$$
\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle=\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle .
$$

(2) $D_{\mathrm{u}} f(0,1)=\nabla f(0,1) \cdot \mathbf{u}=\langle 0,1\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle=\frac{4}{5}$.
(3) $D_{u} g(0,1)=\nabla g(0,1) \cdot \mathbf{u}=\langle 2,0\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle=\frac{6}{5}$.

## Problem 26(b)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v}=\langle 3,4\rangle$.

## Solution:

(1) The unit vector $\mathbf{u}$ in the direction of $\mathbf{v}=\langle 3,4\rangle$ is:

$$
\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle=\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle .
$$

(2) $D_{\mathrm{u}} f(0,1)=\nabla f(0,1) \cdot \mathbf{u}=\langle 0,1\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle=\frac{4}{5}$.
(3) $D_{u} g(0,1)=\nabla g(0,1) \cdot \mathbf{u}=\langle 2,0\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle=\frac{6}{5}$.
(9) $D_{\mathrm{u}} h(0,1)=\nabla h(0,1) \cdot \mathbf{u}$

## Problem 26(b)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v}=\langle 3,4\rangle$.

## Solution:

(1) The unit vector $\mathbf{u}$ in the direction of $\mathbf{v}=\langle 3,4\rangle$ is:

$$
\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle=\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle .
$$

(2) $D_{\mathrm{u}} f(0,1)=\nabla f(0,1) \cdot \mathbf{u}=\langle 0,1\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle=\frac{4}{5}$.
(3) $D_{\mathrm{u}} g(0,1)=\nabla g(0,1) \cdot \mathbf{u}=\langle 2,0\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle=\frac{6}{5}$.
(9) $D_{\mathrm{u}} h(0,1)=\nabla h(0,1) \cdot \mathbf{u}=\langle 1,0\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle$

## Problem 26(b)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v}=\langle 3,4\rangle$.

## Solution:

(1) The unit vector $\mathbf{u}$ in the direction of $\mathbf{v}=\langle 3,4\rangle$ is:

$$
\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{1}{5}\langle 3,4\rangle=\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle .
$$

(2) $D_{\mathrm{u}} f(0,1)=\nabla f(0,1) \cdot \mathbf{u}=\langle 0,1\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle=\frac{4}{5}$.
(3) $D_{u} g(0,1)=\nabla g(0,1) \cdot \mathbf{u}=\langle 2,0\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle=\frac{6}{5}$.
(9) $D_{\mathrm{u}} h(0,1)=\nabla h(0,1) \cdot \mathbf{u}=\langle 1,0\rangle \cdot\left\langle\frac{3}{5}, \frac{4}{5}\right\rangle=\frac{3}{5}$.

## Problem 26(c)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the maximum rate of change (MRC) at the point $(0,1)$.

## Problem 26(c)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the maximum rate of change (MRC) at the point $(0,1)$.

## Solution:

We know that the maximum rate of change is the length of the gradient of the respective function:

## Problem 26(c)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the maximum rate of change (MRC) at the point $(0,1)$.

## Solution:

We know that the maximum rate of change is the length of the gradient of the respective function:

$$
\operatorname{MRC}(f)=|\nabla f(0,1)|=|\langle 0,1\rangle|=1 ;
$$

## Problem 26(c)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the maximum rate of change (MRC) at the point $(0,1)$.

## Solution:

We know that the maximum rate of change is the length of the gradient of the respective function:

$$
\operatorname{MRC}(f)=|\nabla f(0,1)|=|\langle 0,1\rangle|=1 ;
$$

$$
\operatorname{MRC}(g)=|\nabla g(0,1)|=|\langle 2,0\rangle|=2
$$

## Problem 26(c)

For the functions
(1) $f(x, y)=x^{2} y+y^{3}-y^{2}$,
(2) $g(x, y)=x / y+x y$,
(3) $h(x, y)=\sin \left(x^{2} y\right)+x y^{2}$
find the maximum rate of change (MRC) at the point $(0,1)$.

## Solution:

We know that the maximum rate of change is the length of the gradient of the respective function:

$$
\operatorname{MRC}(f)=|\nabla f(0,1)|=|\langle 0,1\rangle|=1 ;
$$

$$
\operatorname{MRC}(g)=|\nabla g(0,1)|=|\langle 2,0\rangle|=2
$$

$\operatorname{MRC}(h)=|\nabla h(0,1)|=|\langle 1,0\rangle|=1$.

## Problem 27

Find an equation of the tangent plane to the surface $x^{2}+2 y^{2}-z^{2}=5$ at the point $(2,1,1)$.

## Problem 27

Find an equation of the tangent plane to the surface $x^{2}+2 y^{2}-z^{2}=5$ at the point $(2,1,1)$.

## Solution:

- For $\mathbf{F}(x, y, z)=x^{2}+2 y^{2}-z^{2}$, the gradient is:

$$
\nabla \mathbf{F}(x, y, z)=\langle 2 x, 4 y,-2 z\rangle
$$

## Problem 27

Find an equation of the tangent plane to the surface $x^{2}+2 y^{2}-z^{2}=5$ at the point $(2,1,1)$.

## Solution:

- For $\mathbf{F}(x, y, z)=x^{2}+2 y^{2}-z^{2}$, the gradient is:

$$
\nabla \mathbf{F}(x, y, z)=\langle 2 x, 4 y,-2 z\rangle
$$

- At the point $(2,1,1)$, we have $\nabla \mathbf{F}(2,1,1)=\langle 4,4,-2\rangle$, which is the normal vector $\mathbf{n}$ to the tangent plane to the surface $\mathbf{F}(x, y, z)=x^{2}+2 y^{2}-z^{2}=5$ at $(2,1,1)$.


## Problem 27

Find an equation of the tangent plane to the surface $x^{2}+2 y^{2}-z^{2}=5$ at the point $(2,1,1)$.

## Solution:

- For $\mathbf{F}(x, y, z)=x^{2}+2 y^{2}-z^{2}$, the gradient is:

$$
\nabla \mathbf{F}(x, y, z)=\langle 2 x, 4 y,-2 z\rangle
$$

- At the point $(2,1,1)$, we have $\nabla \mathbf{F}(2,1,1)=\langle 4,4,-2\rangle$, which is the normal vector $\mathbf{n}$ to the tangent plane to the surface $\mathbf{F}(x, y, z)=x^{2}+2 y^{2}-z^{2}=5$ at $(2,1,1)$.
- Since $(2,1,1)$ is a point on the tangent plane, the equation is:


## Problem 27

Find an equation of the tangent plane to the surface $x^{2}+2 y^{2}-z^{2}=5$ at the point $(2,1,1)$.

## Solution:

- For $\mathbf{F}(x, y, z)=x^{2}+2 y^{2}-z^{2}$, the gradient is:

$$
\nabla \mathbf{F}(x, y, z)=\langle 2 x, 4 y,-2 z\rangle
$$

- At the point $(2,1,1)$, we have $\nabla \mathbf{F}(2,1,1)=\langle 4,4,-2\rangle$, which is the normal vector $\mathbf{n}$ to the tangent plane to the surface $\mathbf{F}(x, y, z)=x^{2}+2 y^{2}-z^{2}=5$ at $(2,1,1)$.
- Since $(2,1,1)$ is a point on the tangent plane, the equation is:

$$
\langle 4,4,-2\rangle \cdot\langle x-2, y-1, z-1\rangle=4(x-2)+4(y-1)-2(z-1)=0 .
$$

## Problem 28

Find parametric equations for the tangent line to the curve of intersection of the surfaces $z^{2}=x^{2}+y^{2}$ and $x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$.

## Problem 28

Find parametric equations for the tangent line to the curve of intersection of the surfaces $z^{2}=x^{2}+y^{2}$ and $x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$.

## Solution:

- If $\mathbf{n}_{1}, \mathbf{n}_{2}$ are the normal vectors of the respective surfaces, the equation of the tangent line is $\mathbf{L}(t)=\langle 3,4,5\rangle+t\left(\mathbf{n}_{1} \times \mathbf{n}_{2}\right)$.


## Problem 28

Find parametric equations for the tangent line to the curve of intersection of the surfaces $z^{2}=x^{2}+y^{2}$ and $x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$.

## Solution:

- If $\mathbf{n}_{1}, \mathbf{n}_{2}$ are the normal vectors of the respective surfaces, the equation of the tangent line is $\mathbf{L}(t)=\langle 3,4,5\rangle+t\left(\mathbf{n}_{1} \times \mathbf{n}_{2}\right)$.
- The normal $\mathbf{n}_{1}$ to the surface $\mathbf{F}(x, y, z)=z^{2}-x^{2}-y^{2}=0$ at the point $(3,4,5)$ is:


## Problem 28

Find parametric equations for the tangent line to the curve of intersection of the surfaces $z^{2}=x^{2}+y^{2}$ and $x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$.

## Solution:

- If $\mathbf{n}_{1}, \mathbf{n}_{2}$ are the normal vectors of the respective surfaces, the equation of the tangent line is $\mathbf{L}(t)=\langle 3,4,5\rangle+t\left(\mathbf{n}_{1} \times \mathbf{n}_{2}\right)$.
- The normal $\mathbf{n}_{1}$ to the surface $\mathbf{F}(x, y, z)=z^{2}-x^{2}-y^{2}=0$ at the point $(3,4,5)$ is: $\quad \mathbf{n}_{1}=\nabla \mathbf{F}(3,4,5)$


## Problem 28

Find parametric equations for the tangent line to the curve of intersection of the surfaces $z^{2}=x^{2}+y^{2}$ and $x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$.

## Solution:

- If $\mathbf{n}_{1}, \mathbf{n}_{2}$ are the normal vectors of the respective surfaces, the equation of the tangent line is $\mathbf{L}(t)=\langle 3,4,5\rangle+t\left(\mathbf{n}_{1} \times \mathbf{n}_{2}\right)$.
- The normal $\mathbf{n}_{1}$ to the surface $\mathbf{F}(x, y, z)=z^{2}-x^{2}-y^{2}=0$ at the point $(3,4,5)$ is: $\quad \mathbf{n}_{1}=\nabla \mathbf{F}(3,4,5)=\langle-6,-8,10\rangle$.


## Problem 28

Find parametric equations for the tangent line to the curve of intersection of the surfaces $z^{2}=x^{2}+y^{2}$ and $x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$.

## Solution:

- If $\mathbf{n}_{1}, \mathbf{n}_{2}$ are the normal vectors of the respective surfaces, the equation of the tangent line is $\mathbf{L}(t)=\langle 3,4,5\rangle+t\left(\mathbf{n}_{1} \times \mathbf{n}_{2}\right)$.
- The normal $\mathbf{n}_{1}$ to the surface $\mathbf{F}(x, y, z)=z^{2}-x^{2}-y^{2}=0$ at the point $(3,4,5)$ is: $\quad \mathbf{n}_{1}=\nabla \mathbf{F}(3,4,5)=\langle-6,-8,10\rangle$.
- The normal $\mathbf{n}_{2}$ to the surface $\mathbf{G}(x, y, z)=x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$ is:


## Problem 28

Find parametric equations for the tangent line to the curve of intersection of the surfaces $z^{2}=x^{2}+y^{2}$ and $x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$.

## Solution:

- If $\mathbf{n}_{1}, \mathbf{n}_{2}$ are the normal vectors of the respective surfaces, the equation of the tangent line is $\mathbf{L}(t)=\langle 3,4,5\rangle+t\left(\mathbf{n}_{1} \times \mathbf{n}_{2}\right)$.
- The normal $\mathbf{n}_{1}$ to the surface $\mathbf{F}(x, y, z)=z^{2}-x^{2}-y^{2}=0$ at the point $(3,4,5)$ is: $\quad \mathbf{n}_{1}=\nabla \mathbf{F}(3,4,5)=\langle-6,-8,10\rangle$.
- The normal $\mathbf{n}_{2}$ to the surface $\mathbf{G}(x, y, z)=x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$ is: $\quad \mathbf{n}_{2}=\nabla \mathbf{G}(3,4,5)$


## Problem 28

Find parametric equations for the tangent line to the curve of intersection of the surfaces $z^{2}=x^{2}+y^{2}$ and $x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$.

## Solution:

- If $\mathbf{n}_{1}, \mathbf{n}_{2}$ are the normal vectors of the respective surfaces, the equation of the tangent line is $\mathbf{L}(t)=\langle 3,4,5\rangle+t\left(\mathbf{n}_{1} \times \mathbf{n}_{2}\right)$.
- The normal $\mathbf{n}_{1}$ to the surface $\mathbf{F}(x, y, z)=z^{2}-x^{2}-y^{2}=0$ at the point $(3,4,5)$ is: $\quad \mathbf{n}_{1}=\nabla \mathbf{F}(3,4,5)=\langle-6,-8,10\rangle$.
- The normal $\mathbf{n}_{2}$ to the surface $\mathbf{G}(x, y, z)=x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$ is: $\quad \mathbf{n}_{2}=\nabla \mathbf{G}(3,4,5)=\langle 6,16,10\rangle$.


## Problem 28

Find parametric equations for the tangent line to the curve of intersection of the surfaces $z^{2}=x^{2}+y^{2}$ and $x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$.

## Solution:

- If $\mathbf{n}_{1}, \mathbf{n}_{2}$ are the normal vectors of the respective surfaces, the equation of the tangent line is $\mathbf{L}(t)=\langle 3,4,5\rangle+t\left(\mathbf{n}_{1} \times \mathbf{n}_{2}\right)$.
- The normal $\mathbf{n}_{1}$ to the surface $\mathbf{F}(x, y, z)=z^{2}-x^{2}-y^{2}=0$ at the point $(3,4,5)$ is: $\quad \mathbf{n}_{1}=\nabla \mathbf{F}(3,4,5)=\langle-6,-8,10\rangle$.
- The normal $\mathbf{n}_{2}$ to the surface $\mathbf{G}(x, y, z)=x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$ is: $\quad \mathbf{n}_{2}=\nabla \mathbf{G}(3,4,5)=\langle 6,16,10\rangle$.
- The vector part of the line is:

$$
\mathbf{n}_{1} \times \mathbf{n}_{2}=\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-6 & -8 & 10 \\
6 & 16 & 10
\end{array}\right|
$$

## Problem 28

Find parametric equations for the tangent line to the curve of intersection of the surfaces $z^{2}=x^{2}+y^{2}$ and $x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$.

## Solution:

- If $\mathbf{n}_{1}, \mathbf{n}_{2}$ are the normal vectors of the respective surfaces, the equation of the tangent line is $\mathbf{L}(t)=\langle 3,4,5\rangle+t\left(\mathbf{n}_{1} \times \mathbf{n}_{2}\right)$.
- The normal $\mathbf{n}_{1}$ to the surface $\mathbf{F}(x, y, z)=z^{2}-x^{2}-y^{2}=0$ at the point $(3,4,5)$ is: $\quad \mathbf{n}_{1}=\nabla \mathbf{F}(3,4,5)=\langle-6,-8,10\rangle$.
- The normal $\mathbf{n}_{2}$ to the surface $\mathbf{G}(x, y, z)=x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$ is: $\quad \mathbf{n}_{2}=\nabla \mathbf{G}(3,4,5)=\langle 6,16,10\rangle$.
- The vector part of the line is:

$$
\mathbf{n}_{1} \times \mathbf{n}_{2}=\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-6 & -8 & 10 \\
6 & 16 & 10
\end{array}\right|=\langle-240,120,-48\rangle
$$

## Problem 28

Find parametric equations for the tangent line to the curve of intersection of the surfaces $z^{2}=x^{2}+y^{2}$ and $x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$.

## Solution:

- If $\mathbf{n}_{1}, \mathbf{n}_{2}$ are the normal vectors of the respective surfaces, the equation of the tangent line is $\mathbf{L}(t)=\langle 3,4,5\rangle+t\left(\mathbf{n}_{1} \times \mathbf{n}_{2}\right)$.
- The normal $\mathbf{n}_{1}$ to the surface $\mathbf{F}(x, y, z)=z^{2}-x^{2}-y^{2}=0$ at the point $(3,4,5)$ is: $\quad \mathbf{n}_{1}=\nabla \mathbf{F}(3,4,5)=\langle-6,-8,10\rangle$.
- The normal $\mathbf{n}_{2}$ to the surface $\mathbf{G}(x, y, z)=x^{2}+2 y^{2}+z^{2}=66$ at the point $(3,4,5)$ is: $\quad \mathbf{n}_{2}=\nabla \mathbf{G}(3,4,5)=\langle 6,16,10\rangle$.
- The vector part of the line is:

$$
\mathbf{n}_{1} \times \mathbf{n}_{2}=\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-6 & -8 & 10 \\
6 & 16 & 10
\end{array}\right|=\langle-240,120,-48\rangle
$$

- The parametric equations are:

$$
\begin{gathered}
x=3-240 t \\
y=4+120 t \\
z=5-48 t .
\end{gathered}
$$

## Problem 29(1)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function
$f(x, y)=x^{2} y^{2}+x^{2}-2 y^{3}+3 y^{2}$

## Problem 29(1)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function
$f(x, y)=x^{2} y^{2}+x^{2}-2 y^{3}+3 y^{2}$
Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\nabla f=\left\langle 2 x y^{2}+2 x, 2 x^{2} y-6 y^{2}+6 y\right\rangle=\langle 0,0\rangle
$$

## Problem 29(1)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function

$$
f(x, y)=x^{2} y^{2}+x^{2}-2 y^{3}+3 y^{2}
$$

Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\begin{aligned}
& \nabla f=\left\langle 2 x y^{2}+2 x, 2 x^{2} y-6 y^{2}+6 y\right\rangle=\langle 0,0\rangle \Longrightarrow \\
& 2 x y+2 x=2 x\left(y^{2}+1\right)=0
\end{aligned}
$$

## Problem 29(1)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function

$$
f(x, y)=x^{2} y^{2}+x^{2}-2 y^{3}+3 y^{2}
$$

Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\begin{aligned}
& \nabla f=\left\langle 2 x y^{2}+2 x, 2 x^{2} y-6 y^{2}+6 y\right\rangle=\langle 0,0\rangle \Longrightarrow \\
& 2 x y+2 x=2 x\left(y^{2}+1\right)=0 \Longrightarrow x=0
\end{aligned}
$$

## Problem 29(1)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function

$$
f(x, y)=x^{2} y^{2}+x^{2}-2 y^{3}+3 y^{2}
$$

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\begin{aligned}
& \nabla f=\left\langle 2 x y^{2}+2 x, 2 x^{2} y-6 y^{2}+6 y\right\rangle=\langle 0,0\rangle \Longrightarrow \\
& 2 x y+2 x=2 x\left(y^{2}+1\right)=0 \Longrightarrow x=0 ; \Longrightarrow \\
& -6 y^{2}+6 y=6 y(-y+1)=0
\end{aligned}
$$

## Problem 29(1)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function

$$
f(x, y)=x^{2} y^{2}+x^{2}-2 y^{3}+3 y^{2}
$$

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\begin{aligned}
& \nabla f=\left\langle 2 x y^{2}+2 x, 2 x^{2} y-6 y^{2}+6 y\right\rangle=\langle 0,0\rangle \Longrightarrow \\
& 2 x y+2 x=2 x\left(y^{2}+1\right)=0 \Longrightarrow x=0 ; \Longrightarrow \\
& -6 y^{2}+6 y=6 y(-y+1)=0 \Longrightarrow y=0 \text { or } y=1
\end{aligned}
$$

## Problem 29(1)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function
$f(x, y)=x^{2} y^{2}+x^{2}-2 y^{3}+3 y^{2}$

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\begin{aligned}
& \nabla f=\left\langle 2 x y^{2}+2 x, 2 x^{2} y-6 y^{2}+6 y\right\rangle=\langle 0,0\rangle \Longrightarrow \\
& 2 x y+2 x=2 x\left(y^{2}+1\right)=0 \Longrightarrow x=0 ; \Longrightarrow \\
& -6 y^{2}+6 y=6 y(-y+1)=0 \Longrightarrow y=0 \text { or } y=1
\end{aligned}
$$

- The critical points are $(0,0),(0,1)$.


## Problem 29(1)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function
$f(x, y)=x^{2} y^{2}+x^{2}-2 y^{3}+3 y^{2}$

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\begin{aligned}
& \nabla f=\left\langle 2 x y^{2}+2 x, 2 x^{2} y-6 y^{2}+6 y\right\rangle=\langle 0,0\rangle \Longrightarrow \\
& 2 x y+2 x=2 x\left(y^{2}+1\right)=0 \Longrightarrow x=0 ; \Longrightarrow \\
& -6 y^{2}+6 y=6 y(-y+1)=0 \Longrightarrow y=0 \text { or } y=1 .
\end{aligned}
$$

- The critical points are $(0,0),(0,1)$.
- The Hessian is:

$$
D=\left|\begin{array}{cc}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
2 y^{2}+2 & 4 x y \\
4 x y & 2 x^{2}-12 y+6
\end{array}\right|
$$

## Problem 29(1)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function
$f(x, y)=x^{2} y^{2}+x^{2}-2 y^{3}+3 y^{2}$

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\begin{aligned}
& \nabla f=\left\langle 2 x y^{2}+2 x, 2 x^{2} y-6 y^{2}+6 y\right\rangle=\langle 0,0\rangle \Longrightarrow \\
& 2 x y+2 x=2 x\left(y^{2}+1\right)=0 \Longrightarrow x=0 ; \Longrightarrow \\
& -6 y^{2}+6 y=6 y(-y+1)=0 \Longrightarrow y=0 \text { or } y=1 .
\end{aligned}
$$

- The critical points are $(0,0),(0,1)$.
- The Hessian is:

$$
D=\left|\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
2 y^{2}+2 & 4 x y \\
4 x y & 2 x^{2}-12 y+6
\end{array}\right|
$$

- Since $D(0,0)=2 \cdot 6>0$ and $f_{x x}(0,0)=2>0$, the point $(0,0)$ is a local minimum.


## Problem 29(1)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function

$$
f(x, y)=x^{2} y^{2}+x^{2}-2 y^{3}+3 y^{2}
$$

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\begin{aligned}
& \nabla f=\left\langle 2 x y^{2}+2 x, 2 x^{2} y-6 y^{2}+6 y\right\rangle=\langle 0,0\rangle \Longrightarrow \\
& 2 x y+2 x=2 x\left(y^{2}+1\right)=0 \Longrightarrow x=0 ; \Longrightarrow \\
& -6 y^{2}+6 y=6 y(-y+1)=0 \Longrightarrow y=0 \text { or } y=1 .
\end{aligned}
$$

- The critical points are $(0,0),(0,1)$.
- The Hessian is:

$$
D=\left|\begin{array}{cc}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right|=\left|\begin{array}{cc}
2 y^{2}+2 & 4 x y \\
4 x y & 2 x^{2}-12 y+6
\end{array}\right| .
$$

- Since $D(0,0)=2 \cdot 6>0$ and $f_{x x}(0,0)=2>0$, the point $(0,0)$ is a local minimum.
- Since $D(0,1)=4 \cdot(-12)<0$, the point $(0,1)$ is saddle point.


## Problem 29(2)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function $g(x, y)=x^{3}+y^{2}+2 x y-4 x-3 y+5$.

## Problem 29(2)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function $g(x, y)=x^{3}+y^{2}+2 x y-4 x-3 y+5$.

## Solution:

- Set $\nabla g=\langle 0,0\rangle$ and solve:

$$
\nabla g=\left\langle 3 x^{2}+2 y-4,2 y+2 x-3\right\rangle=0
$$

## Problem 29(2)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function $g(x, y)=x^{3}+y^{2}+2 x y-4 x-3 y+5$.

## Solution:

- Set $\nabla g=\langle 0,0\rangle$ and solve:

$$
\begin{aligned}
& \nabla g=\left\langle 3 x^{2}+2 y-4,2 y+2 x-3\right\rangle=0 \\
& \quad 2 y+2 x-3=0
\end{aligned}
$$

## Problem 29(2)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function $g(x, y)=x^{3}+y^{2}+2 x y-4 x-3 y+5$.

## Solution:

- Set $\nabla g=\langle 0,0\rangle$ and solve:

$$
\begin{gathered}
\nabla g=\left\langle 3 x^{2}+2 y-4,2 y+2 x-3\right\rangle=0 \\
2 y+2 x-3=0 \Longrightarrow y=\frac{3}{2}-x
\end{gathered}
$$

## Problem 29(2)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function $g(x, y)=x^{3}+y^{2}+2 x y-4 x-3 y+5$.

## Solution:

- Set $\nabla g=\langle 0,0\rangle$ and solve:
- So,

$$
\begin{gathered}
\nabla g=\left\langle 3 x^{2}+2 y-4,2 y+2 x-3\right\rangle=0 \\
2 y+2 x-3=0 \Longrightarrow y=\frac{3}{2}-x
\end{gathered}
$$

$$
3 x^{2}+2\left(\frac{3}{2}-x\right)-4
$$

## Problem 29(2)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function $g(x, y)=x^{3}+y^{2}+2 x y-4 x-3 y+5$.

## Solution:

- Set $\nabla g=\langle 0,0\rangle$ and solve:
- So,

$$
\begin{gathered}
\nabla g=\left\langle 3 x^{2}+2 y-4,2 y+2 x-3\right\rangle=0 \\
2 y+2 x-3=0 \Longrightarrow y=\frac{3}{2}-x
\end{gathered}
$$

$$
3 x^{2}+2\left(\frac{3}{2}-x\right)-4=3 x^{2}-2 x-1
$$

## Problem 29(2)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function $g(x, y)=x^{3}+y^{2}+2 x y-4 x-3 y+5$.

## Solution:

- Set $\nabla g=\langle 0,0\rangle$ and solve:
- So,

$$
\begin{gathered}
\nabla g=\left\langle 3 x^{2}+2 y-4,2 y+2 x-3\right\rangle=0 \\
2 y+2 x-3=0 \Longrightarrow y=\frac{3}{2}-x
\end{gathered}
$$

$$
3 x^{2}+2\left(\frac{3}{2}-x\right)-4=3 x^{2}-2 x-1=(3 x+1)(x-1)=0
$$

## Problem 29(2)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function $g(x, y)=x^{3}+y^{2}+2 x y-4 x-3 y+5$.

## Solution:

- Set $\nabla g=\langle 0,0\rangle$ and solve:
- So,

$$
\begin{gathered}
\nabla g=\left\langle 3 x^{2}+2 y-4,2 y+2 x-3\right\rangle=0 \\
2 y+2 x-3=0 \Longrightarrow y=\frac{3}{2}-x
\end{gathered}
$$

$$
\begin{aligned}
& 3 x^{2}+2\left(\frac{3}{2}-x\right)-4=3 x^{2}-2 x-1=(3 x+1)(x-1)=0 \\
& \Longrightarrow x=1 \text { or } x=-\frac{1}{3}
\end{aligned}
$$

## Problem 29(2)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function $g(x, y)=x^{3}+y^{2}+2 x y-4 x-3 y+5$.

## Solution:

- Set $\nabla g=\langle 0,0\rangle$ and solve:
- So,

$$
\begin{gathered}
\nabla g=\left\langle 3 x^{2}+2 y-4,2 y+2 x-3\right\rangle=0 \\
2 y+2 x-3=0 \Longrightarrow y=\frac{3}{2}-x
\end{gathered}
$$

$$
\begin{aligned}
& 3 x^{2}+2\left(\frac{3}{2}-x\right)-4=3 x^{2}-2 x-1=(3 x+1)(x-1)=0 \\
& \Longrightarrow x=1 \text { or } x=-\frac{1}{3}
\end{aligned}
$$

- The set of critical points is $\left\{\left(1, \frac{1}{2}\right),\left(-\frac{1}{3}, \frac{11}{6}\right)\right\}$.


## Problem 29(2)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function $g(x, y)=x^{3}+y^{2}+2 x y-4 x-3 y+5$.

## Solution:

- Set $\nabla g=\langle 0,0\rangle$ and solve:
- So,

$$
\begin{gathered}
\nabla g=\left\langle 3 x^{2}+2 y-4,2 y+2 x-3\right\rangle=0 \\
2 y+2 x-3=0 \Longrightarrow y=\frac{3}{2}-x
\end{gathered}
$$

$$
\begin{aligned}
& 3 x^{2}+2\left(\frac{3}{2}-x\right)-4=3 x^{2}-2 x-1=(3 x+1)(x-1)=0 \\
& \Longrightarrow x=1 \text { or } x=-\frac{1}{3}
\end{aligned}
$$

- The set of critical points is $\left\{\left(1, \frac{1}{2}\right),\left(-\frac{1}{3}, \frac{11}{6}\right)\right\}$.
- The Hessian is:

$$
D=\left|\begin{array}{ll}
g_{x x} & g_{x y} \\
g_{y x} & g_{y y}
\end{array}\right|=\left|\begin{array}{cc}
6 x & 2 \\
2 & 2
\end{array}\right| .
$$

## Problem 29(2)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function $g(x, y)=x^{3}+y^{2}+2 x y-4 x-3 y+5$.

## Solution:

- Set $\nabla g=\langle 0,0\rangle$ and solve:
- So,

$$
\begin{gathered}
\nabla g=\left\langle 3 x^{2}+2 y-4,2 y+2 x-3\right\rangle=0 \\
2 y+2 x-3=0 \Longrightarrow y=\frac{3}{2}-x
\end{gathered}
$$

$$
\begin{aligned}
& 3 x^{2}+2\left(\frac{3}{2}-x\right)-4=3 x^{2}-2 x-1=(3 x+1)(x-1)=0 \\
& \Longrightarrow x=1 \text { or } x=-\frac{1}{3}
\end{aligned}
$$

- The set of critical points is $\left\{\left(1, \frac{1}{2}\right),\left(-\frac{1}{3}, \frac{11}{6}\right)\right\}$.
- The Hessian is:

$$
D=\left|\begin{array}{ll}
g_{x x} & g_{x y} \\
g_{y x} & g_{y y}
\end{array}\right|=\left|\begin{array}{cc}
6 x & 2 \\
2 & 2
\end{array}\right| .
$$

- Since $D\left(1, \frac{1}{2}\right)=(6 \cdot 2-4)>0$ and $g_{x x}\left(1, \frac{1}{2}\right)=6>0$, the point ( $1, \frac{1}{2}$ ) is a local minimum.


## Problem 29(2)

Find and classify all critical points (as local maxima, local minima, or saddle points) of the function $g(x, y)=x^{3}+y^{2}+2 x y-4 x-3 y+5$.

## Solution:

- Set $\nabla g=\langle 0,0\rangle$ and solve:
- So,

$$
\begin{gathered}
\nabla g=\left\langle 3 x^{2}+2 y-4,2 y+2 x-3\right\rangle=0 \\
2 y+2 x-3=0 \Longrightarrow y=\frac{3}{2}-x
\end{gathered}
$$

$$
\begin{aligned}
& 3 x^{2}+2\left(\frac{3}{2}-x\right)-4=3 x^{2}-2 x-1=(3 x+1)(x-1)=0 \\
& \Longrightarrow x=1 \text { or } x=-\frac{1}{3}
\end{aligned}
$$

- The set of critical points is $\left\{\left(1, \frac{1}{2}\right),\left(-\frac{1}{3}, \frac{11}{6}\right)\right\}$.
- The Hessian is:

$$
D=\left|\begin{array}{ll}
g_{x x} & g_{x y} \\
g_{y x} & g_{y y}
\end{array}\right|=\left|\begin{array}{cc}
6 x & 2 \\
2 & 2
\end{array}\right| .
$$

- Since $D\left(1, \frac{1}{2}\right)=(6 \cdot 2-4)>0$ and $g_{x x}\left(1, \frac{1}{2}\right)=6>0$, the point $\left(1, \frac{1}{2}\right)$ is a local minimum.
- Since $D\left(-\frac{1}{3}, \frac{11}{6}\right)=-8<0$, then $\left(-1, \frac{5}{2}\right)$ is a saddle point.


## Problem 30

Find the minimum value of $f(x, y)=3+x y-x-2 y$ on the closed triangular region with vertices $(0,0),(2,0)$ and $(0,3)$.

## Problem 30

Find the minimum value of $f(x, y)=3+x y-x-2 y$ on the closed triangular region with vertices $(0,0),(2,0)$ and $(0,3)$.

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\nabla f=\langle y-1, x-2\rangle=\langle 0,0\rangle \Longrightarrow y=1 \text { and } x=2
$$

## Problem 30

Find the minimum value of $f(x, y)=3+x y-x-2 y$ on the closed triangular region with vertices $(0,0),(2,0)$ and $(0,3)$.

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\nabla f=\langle y-1, x-2\rangle=\langle 0,0\rangle \Longrightarrow y=1 \text { and } x=2
$$

- There is exactly one critical point which is $(2,1)$, but this point is not inside the triangle, so ignore it.


## Problem 30

Find the minimum value of $f(x, y)=3+x y-x-2 y$ on the closed triangular region with vertices $(0,0),(2,0)$ and $(0,3)$.

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\nabla f=\langle y-1, x-2\rangle=\langle 0,0\rangle \Longrightarrow y=1 \text { and } x=2
$$

- There is exactly one critical point which is $(2,1)$, but this point is not inside the triangle, so ignore it.
- On the interval $(0,0)$ to $(2,0)$,

$$
f(x, 0)=3+x \cdot 0-x-2 \cdot 0
$$

## Problem 30

Find the minimum value of $f(x, y)=3+x y-x-2 y$ on the closed triangular region with vertices $(0,0),(2,0)$ and $(0,3)$.

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\nabla f=\langle y-1, x-2\rangle=\langle 0,0\rangle \Longrightarrow y=1 \text { and } x=2
$$

- There is exactly one critical point which is $(2,1)$, but this point is not inside the triangle, so ignore it.
- On the interval $(0,0)$ to $(2,0)$,

$$
f(x, 0)=3+x \cdot 0-x-2 \cdot 0=3-x
$$

## Problem 30

Find the minimum value of $f(x, y)=3+x y-x-2 y$ on the closed triangular region with vertices $(0,0),(2,0)$ and $(0,3)$.

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\nabla f=\langle y-1, x-2\rangle=\langle 0,0\rangle \Longrightarrow y=1 \text { and } x=2
$$

- There is exactly one critical point which is $(2,1)$, but this point is not inside the triangle, so ignore it.
- On the interval $(0,0)$ to $(2,0)$, $f(x, 0)=3+x \cdot 0-x-2 \cdot 0=3-x$, which has a minimum value of 1 at the point $(2,0)$.


## Problem 30

Find the minimum value of $f(x, y)=3+x y-x-2 y$ on the closed triangular region with vertices $(0,0),(2,0)$ and $(0,3)$.

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\nabla f=\langle y-1, x-2\rangle=\langle 0,0\rangle \Longrightarrow y=1 \text { and } x=2
$$

- There is exactly one critical point which is $(2,1)$, but this point is not inside the triangle, so ignore it.
- On the interval $(0,0)$ to $(2,0)$, $f(x, 0)=3+x \cdot 0-x-2 \cdot 0=3-x$, which has a minimum value of 1 at the point $(2,0)$.
- On the interval $(0,0)$ to $(0,3)$,

$$
f(0, y)=3+0 \cdot y-0-2 y
$$

## Problem 30

Find the minimum value of $f(x, y)=3+x y-x-2 y$ on the closed triangular region with vertices $(0,0),(2,0)$ and $(0,3)$.

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\nabla f=\langle y-1, x-2\rangle=\langle 0,0\rangle \Longrightarrow y=1 \text { and } x=2
$$

- There is exactly one critical point which is $(2,1)$, but this point is not inside the triangle, so ignore it.
- On the interval $(0,0)$ to $(2,0)$, $f(x, 0)=3+x \cdot 0-x-2 \cdot 0=3-x$, which has a minimum value of 1 at the point $(2,0)$.
- On the interval $(0,0)$ to $(0,3)$, $f(0, y)=3+0 \cdot y-0-2 y=3-2 y$, which has a minimum value of -3 at $(0,3)$.


## Problem 30

Find the minimum value of $f(x, y)=3+x y-x-2 y$ on the closed triangular region with vertices $(0,0),(2,0)$ and $(0,3)$.

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\nabla f=\langle y-1, x-2\rangle=\langle 0,0\rangle \Longrightarrow y=1 \text { and } x=2
$$

- There is exactly one critical point which is $(2,1)$, but this point is not inside the triangle, so ignore it.
- On the interval $(0,0)$ to $(2,0)$, $f(x, 0)=3+x \cdot 0-x-2 \cdot 0=3-x$, which has a minimum value of 1 at the point $(2,0)$.
- On the interval $(0,0)$ to $(0,3)$, $f(0, y)=3+0 \cdot y-0-2 y=3-2 y$, which has a minimum value of -3 at $(0,3)$.
- On the line segment from $(2,0)$ to $(0,3), y=-\frac{3}{2} x+3$


## Problem 30

Find the minimum value of $f(x, y)=3+x y-x-2 y$ on the closed triangular region with vertices $(0,0),(2,0)$ and $(0,3)$.

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\nabla f=\langle y-1, x-2\rangle=\langle 0,0\rangle \Longrightarrow y=1 \text { and } x=2
$$

- There is exactly one critical point which is $(2,1)$, but this point is not inside the triangle, so ignore it.
- On the interval $(0,0)$ to $(2,0)$, $f(x, 0)=3+x \cdot 0-x-2 \cdot 0=3-x$, which has a minimum value of 1 at the point $(2,0)$.
- On the interval $(0,0)$ to $(0,3)$, $f(0, y)=3+0 \cdot y-0-2 y=3-2 y$, which has a minimum value of -3 at $(0,3)$.
- On the line segment from $(2,0)$ to $(0,3), y=-\frac{3}{2} x+3$ and $f\left(x, y=-\frac{3}{2} x+3\right)$


## Problem 30

Find the minimum value of $f(x, y)=3+x y-x-2 y$ on the closed triangular region with vertices $(0,0),(2,0)$ and $(0,3)$.

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\nabla f=\langle y-1, x-2\rangle=\langle 0,0\rangle \Longrightarrow y=1 \text { and } x=2
$$

- There is exactly one critical point which is $(2,1)$, but this point is not inside the triangle, so ignore it.
- On the interval $(0,0)$ to $(2,0)$, $f(x, 0)=3+x \cdot 0-x-2 \cdot 0=3-x$, which has a minimum value of 1 at the point $(2,0)$.
- On the interval $(0,0)$ to $(0,3)$, $f(0, y)=3+0 \cdot y-0-2 y=3-2 y$, which has a minimum value of -3 at $(0,3)$.
- On the line segment from $(2,0)$ to $(0,3), y=-\frac{3}{2} x+3$ and $f\left(x, y=-\frac{3}{2} x+3\right)=3+x\left(-\frac{3}{2} x+3\right)-x-2\left(-\frac{3}{2} x+3\right)$


## Problem 30

Find the minimum value of $f(x, y)=3+x y-x-2 y$ on the closed triangular region with vertices $(0,0),(2,0)$ and $(0,3)$.

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\nabla f=\langle y-1, x-2\rangle=\langle 0,0\rangle \Longrightarrow y=1 \text { and } x=2
$$

- There is exactly one critical point which is $(2,1)$, but this point is not inside the triangle, so ignore it.
- On the interval $(0,0)$ to $(2,0)$, $f(x, 0)=3+x \cdot 0-x-2 \cdot 0=3-x$, which has a minimum value of 1 at the point $(2,0)$.
- On the interval $(0,0)$ to $(0,3)$, $f(0, y)=3+0 \cdot y-0-2 y=3-2 y$, which has a minimum value of -3 at $(0,3)$.
- On the line segment from $(2,0)$ to $(0,3), y=-\frac{3}{2} x+3$ and $f\left(x, y=-\frac{3}{2} x+3\right)=3+x\left(-\frac{3}{2} x+3\right)-x-2\left(-\frac{3}{2} x+3\right)=$ $-\frac{3}{2} x^{2}+5 x-3$,


## Problem 30

Find the minimum value of $f(x, y)=3+x y-x-2 y$ on the closed triangular region with vertices $(0,0),(2,0)$ and $(0,3)$.

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\nabla f=\langle y-1, x-2\rangle=\langle 0,0\rangle \Longrightarrow y=1 \text { and } x=2
$$

- There is exactly one critical point which is $(2,1)$, but this point is not inside the triangle, so ignore it.
- On the interval $(0,0)$ to $(2,0)$, $f(x, 0)=3+x \cdot 0-x-2 \cdot 0=3-x$, which has a minimum value of 1 at the point $(2,0)$.
- On the interval $(0,0)$ to $(0,3)$, $f(0, y)=3+0 \cdot y-0-2 y=3-2 y$, which has a minimum value of -3 at $(0,3)$.
- On the line segment from $(2,0)$ to $(0,3), y=-\frac{3}{2} x+3$ and $f\left(x, y=-\frac{3}{2} x+3\right)=3+x\left(-\frac{3}{2} x+3\right)-x-2\left(-\frac{3}{2} x+3\right)=$ $-\frac{3}{2} x^{2}+5 x-3$, which has a minimum of $\frac{25}{6}-3$ at $\left(\frac{5}{3}, \frac{1}{2}\right)$.


## Problem 30

Find the minimum value of $f(x, y)=3+x y-x-2 y$ on the closed triangular region with vertices $(0,0),(2,0)$ and $(0,3)$.

## Solution:

- Set $\nabla f=\langle 0,0\rangle$ and solve:

$$
\nabla f=\langle y-1, x-2\rangle=\langle 0,0\rangle \Longrightarrow y=1 \text { and } x=2
$$

- There is exactly one critical point which is $(2,1)$, but this point is not inside the triangle, so ignore it.
- On the interval $(0,0)$ to $(2,0)$, $f(x, 0)=3+x \cdot 0-x-2 \cdot 0=3-x$, which has a minimum value of 1 at the point $(2,0)$.
- On the interval $(0,0)$ to $(0,3)$, $f(0, y)=3+0 \cdot y-0-2 y=3-2 y$, which has a minimum value of -3 at $(0,3)$.
- On the line segment from $(2,0)$ to $(0,3), y=-\frac{3}{2} x+3$ and $f\left(x, y=-\frac{3}{2} x+3\right)=3+x\left(-\frac{3}{2} x+3\right)-x-2\left(-\frac{3}{2} x+3\right)=$ $-\frac{3}{2} x^{2}+5 x-3$, which has a minimum of $\frac{25}{6}-3$ at $\left(\frac{5}{3}, \frac{1}{2}\right)$.
- Hence, the absolute minimum value of $f(x, y)$ is -3 .


## Problem 31(1)

Use Lagrange multipliers to find the extreme values of $f(x, y)=x y$ with constraint $g(x, y)=x^{2}+2 y^{2}=3$.

## Problem 31(1)

Use Lagrange multipliers to find the extreme values of $f(x, y)=x y$ with constraint $g(x, y)=x^{2}+2 y^{2}=3$.

## Solution:

## Problem 31(1)

Use Lagrange multipliers to find the extreme values of $f(x, y)=x y$ with constraint $g(x, y)=x^{2}+2 y^{2}=3$.

## Solution:

- Set $\nabla f=\langle y, x\rangle=\lambda \nabla g=\lambda\langle 2 x, 4 y\rangle$ and solve:


## Problem 31(1)

Use Lagrange multipliers to find the extreme values of $f(x, y)=x y$ with constraint $g(x, y)=x^{2}+2 y^{2}=3$.

## Solution:

- Set $\nabla f=\langle y, x\rangle=\lambda \nabla g=\lambda\langle 2 x, 4 y\rangle$ and solve:

$$
\begin{aligned}
& y=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=\frac{y}{2 x} . \\
& x=\lambda 4 y \Longrightarrow y=0 \text { or } \lambda=\frac{x}{4 y} .
\end{aligned}
$$

## Problem 31(1)

Use Lagrange multipliers to find the extreme values of $f(x, y)=x y$ with constraint $g(x, y)=x^{2}+2 y^{2}=3$.

## Solution:

- Set $\nabla f=\langle y, x\rangle=\lambda \nabla g=\lambda\langle 2 x, 4 y\rangle$ and solve:

$$
\begin{aligned}
& y=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=\frac{y}{2 x} . \\
& x=\lambda 4 y \Longrightarrow y=0 \text { or } \lambda=\frac{x}{4 y} .
\end{aligned}
$$

- Since $g(x, y)=x^{2}+2 y^{2}=3$, either $x$ or $y$ must be nonzero;


## Problem 31(1)

Use Lagrange multipliers to find the extreme values of $f(x, y)=x y$ with constraint $g(x, y)=x^{2}+2 y^{2}=3$.

## Solution:

- Set $\nabla f=\langle y, x\rangle=\lambda \nabla g=\lambda\langle 2 x, 4 y\rangle$ and solve:

$$
\begin{aligned}
& y=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=\frac{y}{2 x} . \\
& x=\lambda 4 y \Longrightarrow y=0 \text { or } \lambda=\frac{x}{4 y} .
\end{aligned}
$$

- Since $g(x, y)=x^{2}+2 y^{2}=3$, either $x$ or $y$ must be nonzero; the above equations then imply both $x$ and $y$ are nonzero.


## Problem 31(1)

Use Lagrange multipliers to find the extreme values of $f(x, y)=x y$ with constraint $g(x, y)=x^{2}+2 y^{2}=3$.

## Solution:

- Set $\nabla f=\langle y, x\rangle=\lambda \nabla g=\lambda\langle 2 x, 4 y\rangle$ and solve:

$$
\begin{aligned}
& y=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=\frac{y}{2 x} . \\
& x=\lambda 4 y \Longrightarrow y=0 \text { or } \lambda=\frac{x}{4 y} .
\end{aligned}
$$

- Since $g(x, y)=x^{2}+2 y^{2}=3$, either $x$ or $y$ must be nonzero; the above equations then imply both $x$ and $y$ are nonzero.
- Since $x, y$ are both nonzero, then

$$
\frac{y}{2 x}=\frac{x}{4 y}
$$

## Problem 31(1)

Use Lagrange multipliers to find the extreme values of $f(x, y)=x y$ with constraint $g(x, y)=x^{2}+2 y^{2}=3$.

## Solution:

- Set $\nabla f=\langle y, x\rangle=\lambda \nabla g=\lambda\langle 2 x, 4 y\rangle$ and solve:

$$
\begin{aligned}
& y=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=\frac{y}{2 x} . \\
& x=\lambda 4 y \Longrightarrow y=0 \text { or } \lambda=\frac{x}{4 y} .
\end{aligned}
$$

- Since $g(x, y)=x^{2}+2 y^{2}=3$, either $x$ or $y$ must be nonzero; the above equations then imply both $x$ and $y$ are nonzero.
- Since $x, y$ are both nonzero, then

$$
\frac{y}{2 x}=\frac{x}{4 y} \Longrightarrow 4 y^{2}=2 x^{2}
$$

## Problem 31(1)

Use Lagrange multipliers to find the extreme values of $f(x, y)=x y$ with constraint $g(x, y)=x^{2}+2 y^{2}=3$.

## Solution:

- Set $\nabla f=\langle y, x\rangle=\lambda \nabla g=\lambda\langle 2 x, 4 y\rangle$ and solve:

$$
\begin{aligned}
& y=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=\frac{y}{2 x} . \\
& x=\lambda 4 y \Longrightarrow y=0 \text { or } \lambda=\frac{x}{4 y} .
\end{aligned}
$$

- Since $g(x, y)=x^{2}+2 y^{2}=3$, either $x$ or $y$ must be nonzero; the above equations then imply both $x$ and $y$ are nonzero.
- Since $x, y$ are both nonzero, then

$$
\frac{y}{2 x}=\frac{x}{4 y} \Longrightarrow 4 y^{2}=2 x^{2} \Longrightarrow x^{2}=2 y^{2}
$$

## Problem 31(1)

Use Lagrange multipliers to find the extreme values of $f(x, y)=x y$ with constraint $g(x, y)=x^{2}+2 y^{2}=3$.

Solution:

- Set $\nabla f=\langle y, x\rangle=\lambda \nabla g=\lambda\langle 2 x, 4 y\rangle$ and solve:

$$
\begin{aligned}
& y=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=\frac{y}{2 x} . \\
& x=\lambda 4 y \Longrightarrow y=0 \text { or } \lambda=\frac{x}{4 y} .
\end{aligned}
$$

- Since $g(x, y)=x^{2}+2 y^{2}=3$, either $x$ or $y$ must be nonzero; the above equations then imply both $x$ and $y$ are nonzero.
- Since $x, y$ are both nonzero, then

$$
\frac{y}{2 x}=\frac{x}{4 y} \Longrightarrow 4 y^{2}=2 x^{2} \Longrightarrow x^{2}=2 y^{2}
$$

- From the constraint $x^{2}+2 y^{2}=3$, we get $y= \pm \frac{\sqrt{3}}{2}$,


## Problem 31(1)

Use Lagrange multipliers to find the extreme values of $f(x, y)=x y$ with constraint $g(x, y)=x^{2}+2 y^{2}=3$.

Solution:

- Set $\nabla f=\langle y, x\rangle=\lambda \nabla g=\lambda\langle 2 x, 4 y\rangle$ and solve:

$$
\begin{aligned}
& y=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=\frac{y}{2 x} . \\
& x=\lambda 4 y \Longrightarrow y=0 \text { or } \lambda=\frac{x}{4 y} .
\end{aligned}
$$

- Since $g(x, y)=x^{2}+2 y^{2}=3$, either $x$ or $y$ must be nonzero; the above equations then imply both $x$ and $y$ are nonzero.
- Since $x, y$ are both nonzero, then

$$
\frac{y}{2 x}=\frac{x}{4 y} \Longrightarrow 4 y^{2}=2 x^{2} \Longrightarrow x^{2}=2 y^{2}
$$

- From the constraint $x^{2}+2 y^{2}=3$, we get $y= \pm \frac{\sqrt{3}}{2}$, and the 4 possible points $\left( \pm \frac{\sqrt{3}}{\sqrt{2}}, \pm \frac{\sqrt{3}}{2}\right)$ where $f(x, y)$ is extreme.


## Problem 31(1)

Use Lagrange multipliers to find the extreme values of $f(x, y)=x y$ with constraint $g(x, y)=x^{2}+2 y^{2}=3$.

Solution:

- Set $\nabla f=\langle y, x\rangle=\lambda \nabla g=\lambda\langle 2 x, 4 y\rangle$ and solve:

$$
\begin{aligned}
& y=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=\frac{y}{2 x} . \\
& x=\lambda 4 y \Longrightarrow y=0 \text { or } \lambda=\frac{x}{4 y} .
\end{aligned}
$$

- Since $g(x, y)=x^{2}+2 y^{2}=3$, either $x$ or $y$ must be nonzero; the above equations then imply both $x$ and $y$ are nonzero.
- Since $x, y$ are both nonzero, then

$$
\frac{y}{2 x}=\frac{x}{4 y} \Longrightarrow 4 y^{2}=2 x^{2} \Longrightarrow x^{2}=2 y^{2}
$$

- From the constraint $x^{2}+2 y^{2}=3$, we get $y= \pm \frac{\sqrt{3}}{2}$, and the 4 possible points $\left( \pm \frac{\sqrt{3}}{\sqrt{2}}, \pm \frac{\sqrt{3}}{2}\right)$ where $f(x, y)$ is extreme.
- Then $f\left(\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}\right)=f\left(-\frac{\sqrt{3}}{\sqrt{2}},-\frac{\sqrt{3}}{2}\right)=\frac{3}{2 \sqrt{2}}$,


## Problem 31(1)

Use Lagrange multipliers to find the extreme values of $f(x, y)=x y$ with constraint $g(x, y)=x^{2}+2 y^{2}=3$.

Solution:

- Set $\nabla f=\langle y, x\rangle=\lambda \nabla g=\lambda\langle 2 x, 4 y\rangle$ and solve:

$$
\begin{aligned}
& y=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=\frac{y}{2 x} . \\
& x=\lambda 4 y \Longrightarrow y=0 \text { or } \lambda=\frac{x}{4 y} .
\end{aligned}
$$

- Since $g(x, y)=x^{2}+2 y^{2}=3$, either $x$ or $y$ must be nonzero; the above equations then imply both $x$ and $y$ are nonzero.
- Since $x, y$ are both nonzero, then

$$
\frac{y}{2 x}=\frac{x}{4 y} \Longrightarrow 4 y^{2}=2 x^{2} \Longrightarrow x^{2}=2 y^{2}
$$

- From the constraint $x^{2}+2 y^{2}=3$, we get $y= \pm \frac{\sqrt{3}}{2}$, and the 4 possible points $\left( \pm \frac{\sqrt{3}}{\sqrt{2}}, \pm \frac{\sqrt{3}}{2}\right)$ where $f(x, y)$ is extreme.
- Then $f\left(\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}\right)=f\left(-\frac{\sqrt{3}}{\sqrt{2}},-\frac{\sqrt{3}}{2}\right)=\frac{3}{2 \sqrt{2}}$,

$$
f\left(-\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}\right)=f\left(\frac{\sqrt{3}}{\sqrt{2}},-\frac{\sqrt{3}}{2}\right)=-\frac{3}{2 \sqrt{2}} .
$$

## Problem 31(1)

Use Lagrange multipliers to find the extreme values of $f(x, y)=x y$ with constraint $g(x, y)=x^{2}+2 y^{2}=3$.

Solution:

- Set $\nabla f=\langle y, x\rangle=\lambda \nabla g=\lambda\langle 2 x, 4 y\rangle$ and solve:

$$
\begin{aligned}
& y=\lambda 2 x \Longrightarrow x=0 \text { or } \lambda=\frac{y}{2 x} . \\
& x=\lambda 4 y \Longrightarrow y=0 \text { or } \lambda=\frac{x}{4 y} .
\end{aligned}
$$

- Since $g(x, y)=x^{2}+2 y^{2}=3$, either $x$ or $y$ must be nonzero; the above equations then imply both $x$ and $y$ are nonzero.
- Since $x, y$ are both nonzero, then

$$
\frac{y}{2 x}=\frac{x}{4 y} \Longrightarrow 4 y^{2}=2 x^{2} \Longrightarrow x^{2}=2 y^{2}
$$

- From the constraint $x^{2}+2 y^{2}=3$, we get $y= \pm \frac{\sqrt{3}}{2}$, and the 4 possible points $\left( \pm \frac{\sqrt{3}}{\sqrt{2}}, \pm \frac{\sqrt{3}}{2}\right)$ where $f(x, y)$ is extreme.
- Then $f\left(\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}\right)=f\left(-\frac{\sqrt{3}}{\sqrt{2}},-\frac{\sqrt{3}}{2}\right)=\frac{3}{2 \sqrt{2}}$,

$$
f\left(-\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}\right)=f\left(\frac{\sqrt{3}}{\sqrt{2}},-\frac{\sqrt{3}}{2}\right)=-\frac{3}{2 \sqrt{2}} .
$$

- Hence, the extreme values are $\pm \frac{3}{2 \sqrt{2}}$.


## Problem 31(2)

Use Lagrange multipliers to find the extreme values of $g(x, y, z)=x+3 y-2 z$ with constraint $x^{2}+2 y^{2}+z^{2}=5$.

## Problem 31(2)

Use Lagrange multipliers to find the extreme values of $g(x, y, z)=x+3 y-2 z$ with constraint $x^{2}+2 y^{2}+z^{2}=5$.

Solution:
There is no Lagrange multipliers problem in 3 variables on this exam.

Problem 32(1)
Find the iterated integral,

$$
\int_{1}^{4} \int_{0}^{2}(x+\sqrt{y}) d x d y
$$

## Problem 32(1)

Find the iterated integral,

$$
\int_{1}^{4} \int_{0}^{2}(x+\sqrt{y}) d x d y
$$

## Solution:

$$
\int_{1}^{4} \int_{0}^{2}(x+\sqrt{y}) d x d y
$$

## Problem 32(1)

Find the iterated integral,

$$
\int_{1}^{4} \int_{0}^{2}(x+\sqrt{y}) d x d y
$$

## Solution:

$$
\int_{1}^{4} \int_{0}^{2}(x+\sqrt{y}) d x d y=\int_{1}^{4}\left[\frac{x^{2}}{2}+\sqrt{y} x\right]_{0}^{2} d y
$$

## Problem 32(1)

Find the iterated integral,

$$
\int_{1}^{4} \int_{0}^{2}(x+\sqrt{y}) d x d y
$$

## Solution:

$$
\begin{gathered}
\int_{1}^{4} \int_{0}^{2}(x+\sqrt{y}) d x d y=\int_{1}^{4}\left[\frac{x^{2}}{2}+\sqrt{y} x\right]_{0}^{2} d y \\
\quad=\int_{1}^{4}\left(2+2 y^{\frac{1}{2}}\right) d y
\end{gathered}
$$

## Problem 32(1)

Find the iterated integral,

$$
\int_{1}^{4} \int_{0}^{2}(x+\sqrt{y}) d x d y
$$

## Solution:

$$
\begin{gathered}
\int_{1}^{4} \int_{0}^{2}(x+\sqrt{y}) d x d y=\int_{1}^{4}\left[\frac{x^{2}}{2}+\sqrt{y} x\right]_{0}^{2} d y \\
=\int_{1}^{4}\left(2+2 y^{\frac{1}{2}}\right) d y=2 y+\left.\frac{4}{3} y^{\frac{3}{2}}\right|_{1} ^{4}
\end{gathered}
$$

## Problem 32(1)

Find the iterated integral,

$$
\int_{1}^{4} \int_{0}^{2}(x+\sqrt{y}) d x d y
$$

## Solution:

$$
\begin{gathered}
\int_{1}^{4} \int_{0}^{2}(x+\sqrt{y}) d x d y=\int_{1}^{4}\left[\frac{x^{2}}{2}+\sqrt{y} x\right]_{0}^{2} d y \\
=\int_{1}^{4}\left(2+2 y^{\frac{1}{2}}\right) d y=2 y+\left.\frac{4}{3} y^{\frac{3}{2}}\right|_{1} ^{4} \\
=8+\frac{4}{3}(8)-\left(2+\frac{4}{3}\right)
\end{gathered}
$$

Problem 32(2)
Find the iterated integral,

$$
\int_{1}^{2} \int_{0}^{1}(2 x+3 y)^{2} d y d x
$$

Problem 32(2)
Find the iterated integral,

$$
\int_{1}^{2} \int_{0}^{1}(2 x+3 y)^{2} d y d x
$$

## Solution:

$$
\int_{1}^{2} \int_{0}^{1}(2 x+3 y)^{2} d y d x
$$

## Problem 32(2)

Find the iterated integral,

$$
\int_{1}^{2} \int_{0}^{1}(2 x+3 y)^{2} d y d x
$$

## Solution:

$$
\int_{1}^{2} \int_{0}^{1}(2 x+3 y)^{2} d y d x=\int_{1}^{2} \int_{0}^{1} 4 x^{2}+12 x y+9 y^{2} d y d x
$$

## Problem 32(2)

Find the iterated integral,

$$
\int_{1}^{2} \int_{0}^{1}(2 x+3 y)^{2} d y d x
$$

## Solution:

$$
\begin{aligned}
& \int_{1}^{2} \int_{0}^{1}(2 x+3 y)^{2} d y d x=\int_{1}^{2} \int_{0}^{1} 4 x^{2}+12 x y+9 y^{2} d y d x \\
& \quad=\int_{1}^{2}\left[4 x^{2} y+6 x y^{2}+3 y^{3}\right]_{0}^{1} d x
\end{aligned}
$$

## Problem 32(2)

Find the iterated integral,

$$
\int_{1}^{2} \int_{0}^{1}(2 x+3 y)^{2} d y d x
$$

## Solution:

$$
\begin{aligned}
& \int_{1}^{2} \int_{0}^{1}(2 x+3 y)^{2} d y d x=\int_{1}^{2} \int_{0}^{1} 4 x^{2}+12 x y+9 y^{2} d y d x \\
& \quad=\int_{1}^{2}\left[4 x^{2} y+6 x y^{2}+3 y^{3}\right]_{0}^{1} d x=\int_{1}^{2} 4 x^{2}+6 x+3 d x
\end{aligned}
$$

## Problem 32(2)

Find the iterated integral,

$$
\int_{1}^{2} \int_{0}^{1}(2 x+3 y)^{2} d y d x
$$

## Solution:

$$
\begin{aligned}
& \int_{1}^{2} \int_{0}^{1}(2 x+3 y)^{2} d y d x=\int_{1}^{2} \int_{0}^{1} 4 x^{2}+12 x y+9 y^{2} d y d x \\
& =\int_{1}^{2}\left[4 x^{2} y+6 x y^{2}+3 y^{3}\right]_{0}^{1} d x=\int_{1}^{2} 4 x^{2}+6 x+3 d x \\
& =\frac{4}{3} x^{3}+3 x^{2}+\left.3 x\right|_{1} ^{2}
\end{aligned}
$$

## Problem 32(2)

Find the iterated integral,

$$
\int_{1}^{2} \int_{0}^{1}(2 x+3 y)^{2} d y d x
$$

## Solution:

$$
\begin{aligned}
& \int_{1}^{2} \int_{0}^{1}(2 x+3 y)^{2} d y d x=\int_{1}^{2} \int_{0}^{1} 4 x^{2}+12 x y+9 y^{2} d y d x \\
& =\int_{1}^{2}\left[4 x^{2} y+6 x y^{2}+3 y^{3}\right]_{0}^{1} d x=\int_{1}^{2} 4 x^{2}+6 x+3 d x \\
& =\frac{4}{3} x^{3}+3 x^{2}+\left.3 x\right|_{1} ^{2}=\frac{4}{3}(2)^{3}+3(2)^{2}+3(2)-\left(\frac{4}{3}+3+3\right)
\end{aligned}
$$

## Problem 32(3)

Find the iterated integral,

$$
\int_{0}^{1} \int_{x}^{2-x}\left(x^{2}-y\right) d y d x
$$

## Solution:

There is no integration problem on this exam with varying limits of integration (function limits).

## Problem 32(4)

Find the iterated integral,

$$
\int_{0}^{1} \int_{x^{2}}^{1} x^{3} \sin \left(y^{3}\right) d y d x
$$

(Hint: Reverse the order of integration)

## Solution:

There is no integration problem on this exam with varying limits of integration (function limits).

## Problem 33(1)

Evaluate the following double integral.

$$
\iint_{\mathbf{R}} \cos (x+2 y) d A, \quad \mathbf{R}=\{(x, y) \mid 0 \leq x \leq \pi, 0 \leq y \leq \pi / 2\} .
$$

## Problem 33(1)

Evaluate the following double integral.

$$
\iint_{\mathbf{R}} \cos (x+2 y) d A, \quad \mathbf{R}=\{(x, y) \mid 0 \leq x \leq \pi, 0 \leq y \leq \pi / 2\}
$$

## Solution:

- Applying Fubini's Theorem and the fact $\sin (\pi+\theta)=-\sin (\theta)$, we obtain:

$$
\iint_{\mathrm{R}} \cos (x+2 y) d A
$$

## Problem 33(1)

Evaluate the following double integral.

$$
\iint_{\mathbf{R}} \cos (x+2 y) d A, \quad \mathbf{R}=\{(x, y) \mid 0 \leq x \leq \pi, 0 \leq y \leq \pi / 2\}
$$

## Solution:

- Applying Fubini's Theorem and the fact $\sin (\pi+\theta)=-\sin (\theta)$, we obtain:

$$
\iint_{\mathrm{R}} \cos (x+2 y) d A=\int_{0}^{\frac{\pi}{2}} \int_{0}^{\pi} \cos (x+2 y) d x d y
$$

## Problem 33(1)

Evaluate the following double integral.

$$
\iint_{\mathbf{R}} \cos (x+2 y) d A, \quad \mathbf{R}=\{(x, y) \mid 0 \leq x \leq \pi, 0 \leq y \leq \pi / 2\}
$$

## Solution:

- Applying Fubini's Theorem and the fact $\sin (\pi+\theta)=-\sin (\theta)$, we obtain:

$$
\begin{aligned}
& \iint_{\mathrm{R}} \cos (x+2 y) d A=\int_{0}^{\frac{\pi}{2}} \int_{0}^{\pi} \cos (x+2 y) d x d y \\
= & \int_{0}^{\frac{\pi}{2}}[\sin (x+2 y)]_{0}^{\pi} d y
\end{aligned}
$$

## Problem 33(1)

Evaluate the following double integral.

$$
\iint_{\mathbf{R}} \cos (x+2 y) d A, \quad \mathbf{R}=\{(x, y) \mid 0 \leq x \leq \pi, 0 \leq y \leq \pi / 2\}
$$

## Solution:

- Applying Fubini's Theorem and the fact $\sin (\pi+\theta)=-\sin (\theta)$, we obtain:

$$
\begin{aligned}
& \iint_{\mathrm{R}} \cos (x+2 y) d A=\int_{0}^{\frac{\pi}{2}} \int_{0}^{\pi} \cos (x+2 y) d x d y \\
= & \int_{0}^{\frac{\pi}{2}}[\sin (x+2 y)]_{0}^{\pi} d y=\int_{0}^{\frac{\pi}{2}} \sin (\pi+2 y)-\sin (2 y) d y
\end{aligned}
$$

## Problem 33(1)

Evaluate the following double integral.

$$
\iint_{\mathbf{R}} \cos (x+2 y) d A, \quad \mathbf{R}=\{(x, y) \mid 0 \leq x \leq \pi, 0 \leq y \leq \pi / 2\}
$$

## Solution:

- Applying Fubini's Theorem and the fact $\sin (\pi+\theta)=-\sin (\theta)$, we obtain:

$$
\begin{aligned}
& \iint_{\mathrm{R}} \cos (x+2 y) d A=\int_{0}^{\frac{\pi}{2}} \int_{0}^{\pi} \cos (x+2 y) d x d y \\
= & \int_{0}^{\frac{\pi}{2}}[\sin (x+2 y)]_{0}^{\pi} d y=\int_{0}^{\frac{\pi}{2}} \sin (\pi+2 y)-\sin (2 y) d y \\
= & \int_{0}^{\frac{\pi}{2}}-\sin (2 y) 2 d y
\end{aligned}
$$

## Problem 33(1)

Evaluate the following double integral.

$$
\iint_{\mathbf{R}} \cos (x+2 y) d A, \quad \mathbf{R}=\{(x, y) \mid 0 \leq x \leq \pi, 0 \leq y \leq \pi / 2\}
$$

## Solution:

- Applying Fubini's Theorem and the fact $\sin (\pi+\theta)=-\sin (\theta)$, we obtain:

$$
\begin{aligned}
& \iint_{\mathrm{R}} \cos (x+2 y) d A=\int_{0}^{\frac{\pi}{2}} \int_{0}^{\pi} \cos (x+2 y) d x d y \\
= & \int_{0}^{\frac{\pi}{2}}[\sin (x+2 y)]_{0}^{\pi} d y=\int_{0}^{\frac{\pi}{2}} \sin (\pi+2 y)-\sin (2 y) d y \\
= & \int_{0}^{\frac{\pi}{2}}-\sin (2 y) 2 d y=\left.\cos (2 y)\right|_{0} ^{\frac{\pi}{2}}=-1-1=-2 .
\end{aligned}
$$

## Problem 33(2)

Evaluate the following double integral.

$$
\iint_{\mathbf{R}} e^{y^{2}} d A, \quad \mathbf{R}=\{(x, y) \mid 0 \leq y \leq 1,0 \leq x \leq y\} .
$$

## Solution:

There is no integration problem on this exam with varying limits of integration (function limits).

## Problem 33(3)

Evaluate the following double integral.

$$
\iint_{\mathrm{R}} x \sqrt{y^{2}-x^{2}} d A, \quad \mathbf{R}=\{(x, y) \mid 0 \leq y \leq 1,0 \leq x \leq y\} .
$$

Solution:
There is no integration problem on this exam with varying limits of integration (function limits).

## Problem 34(1)

Find the volume $\mathbf{V}$ of the solid under the surface $z=4+x^{2}-y^{2}$ and above the rectangle

$$
\mathbf{R}=\{(x, y) \mid-1 \leq x \leq 1,0 \leq y \leq 2\}
$$

## Problem 34(1)

Find the volume $\mathbf{V}$ of the solid under the surface $z=4+x^{2}-y^{2}$ and above the rectangle

$$
\mathbf{R}=\{(x, y) \mid-1 \leq x \leq 1,0 \leq y \leq 2\}
$$

## Solution:

$$
\mathbf{V}=\iint_{\mathbf{R}}\left(4+x^{2}-y^{2}\right) d A
$$

## Problem 34(1)

Find the volume $\mathbf{V}$ of the solid under the surface $z=4+x^{2}-y^{2}$ and above the rectangle

$$
\mathbf{R}=\{(x, y) \mid-1 \leq x \leq 1,0 \leq y \leq 2\}
$$

## Solution:

$$
\mathbf{V}=\iint_{\mathbf{R}}\left(4+x^{2}-y^{2}\right) d A=\int_{0}^{2} \int_{-1}^{1}\left(4+x^{2}-y^{2}\right) d x d y
$$

## Problem 34(1)

Find the volume $\mathbf{V}$ of the solid under the surface $z=4+x^{2}-y^{2}$ and above the rectangle

$$
\mathbf{R}=\{(x, y) \mid-1 \leq x \leq 1,0 \leq y \leq 2\}
$$

## Solution:

$$
\begin{aligned}
\mathbf{V} & =\iint_{\mathbf{R}}\left(4+x^{2}-y^{2}\right) d A=\int_{0}^{2} \int_{-1}^{1}\left(4+x^{2}-y^{2}\right) d x d y \\
& =\int_{0}^{2}\left[4 x+\frac{x^{3}}{3}-y^{2} x\right]_{-1}^{1} d y
\end{aligned}
$$

## Problem 34(1)

Find the volume $\mathbf{V}$ of the solid under the surface $z=4+x^{2}-y^{2}$ and above the rectangle

$$
\mathbf{R}=\{(x, y) \mid-1 \leq x \leq 1,0 \leq y \leq 2\}
$$

## Solution:

$$
\begin{aligned}
\mathbf{V} & =\iint_{\mathbf{R}}\left(4+x^{2}-y^{2}\right) d A=\int_{0}^{2} \int_{-1}^{1}\left(4+x^{2}-y^{2}\right) d x d y \\
& =\int_{0}^{2}\left[4 x+\frac{x^{3}}{3}-y^{2} x\right]_{-1}^{1} d y=\int_{0}^{2}\left(8+\frac{2}{3}-2 y^{2}\right) d y
\end{aligned}
$$

## Problem 34(1)

Find the volume $\mathbf{V}$ of the solid under the surface $z=4+x^{2}-y^{2}$ and above the rectangle

$$
\mathbf{R}=\{(x, y) \mid-1 \leq x \leq 1,0 \leq y \leq 2\}
$$

## Solution:

$$
\begin{aligned}
& \mathbf{V}=\iint_{R}\left(4+x^{2}-y^{2}\right) d A=\int_{0}^{2} \int_{-1}^{1}\left(4+x^{2}-y^{2}\right) d x d y \\
&=\int_{0}^{2}\left[4 x+\frac{x^{3}}{3}-y^{2} x\right]_{-1}^{1} d y=\int_{0}^{2}\left(8+\frac{2}{3}-2 y^{2}\right) d y \\
&=\left(8+\frac{2}{3}\right) y-\left.\frac{2}{3} y^{3}\right|_{0} ^{2}
\end{aligned}
$$

## Problem 34(1)

Find the volume $\mathbf{V}$ of the solid under the surface $z=4+x^{2}-y^{2}$ and above the rectangle

$$
\mathbf{R}=\{(x, y) \mid-1 \leq x \leq 1,0 \leq y \leq 2\}
$$

## Solution:

$$
\begin{gathered}
\mathbf{V}=\iint_{\mathrm{R}}\left(4+x^{2}-y^{2}\right) d A=\int_{0}^{2} \int_{-1}^{1}\left(4+x^{2}-y^{2}\right) d x d y \\
=\int_{0}^{2}\left[4 x+\frac{x^{3}}{3}-y^{2} x\right]_{-1}^{1} d y=\int_{0}^{2}\left(8+\frac{2}{3}-2 y^{2}\right) d y \\
=\left(8+\frac{2}{3}\right) y-\left.\frac{2}{3} y^{3}\right|_{0} ^{2}=\frac{52}{3}-\frac{16}{3}
\end{gathered}
$$

## Problem 34(1)

Find the volume $\mathbf{V}$ of the solid under the surface $z=4+x^{2}-y^{2}$ and above the rectangle

$$
\mathbf{R}=\{(x, y) \mid-1 \leq x \leq 1,0 \leq y \leq 2\}
$$

## Solution:

$$
\begin{gathered}
\mathbf{V}=\iint_{R}\left(4+x^{2}-y^{2}\right) d A=\int_{0}^{2} \int_{-1}^{1}\left(4+x^{2}-y^{2}\right) d x d y \\
=\int_{0}^{2}\left[4 x+\frac{x^{3}}{3}-y^{2} x\right]_{-1}^{1} d y=\int_{0}^{2}\left(8+\frac{2}{3}-2 y^{2}\right) d y \\
=\left(8+\frac{2}{3}\right) y-\left.\frac{2}{3} y^{3}\right|_{0} ^{2}=\frac{52}{3}-\frac{16}{3}=\frac{36}{3}
\end{gathered}
$$

## Problem 34(2)

Find the volume $\mathbf{V}$ of the solid under the surface $z=2 x+y^{2}$ and above the region bounded by curves $x-y^{2}=0$ and $x-y^{3}=0$.

## Solution:

There is no integration problem on this exam with varying limits of integration (function limits).

