Math 421 – Practice Final Solutions
May 12, 2009

1. Find the Taylor series or Laurent series of $f(z) = \dfrac{2}{z-1}$ in the following domains.

   (a) $D_1 = \{z \mid |z| < 1\}$.

   Solution: Since $|z| < 1$, $f(z) = \dfrac{2}{z-1} = \dfrac{-2}{1-z} = -2\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty}(-2)z^n$.

   (b) $D_2 = \{z \mid 1 < |z| < \infty\}$.

   Solution: Note that $1 < |z| < \infty \Rightarrow |\frac{1}{z}| < 1$, so we have

$$f(z) = \frac{2}{z-1} = \frac{2}{z(1-\frac{1}{z})} = \frac{2}{z}\sum_{n=0}^{\infty}(\frac{1}{z})^n = \sum_{n=0}^{\infty}\frac{2}{z^{n+1}}.$$

   (c) $D_5 = \{z \mid 1 < |z-2| < \infty\}$.

   Solution: Note that $1 < |z-2| < \infty \Rightarrow |\frac{1}{z-2}| < 1$, so we have

$$f(z) = \frac{2}{z-1} = \frac{2}{(z-2)+1} = \frac{2}{(z-2)(1-\frac{-1}{z-2})} = \frac{2}{z-2}\sum_{n=0}^{\infty}(\frac{-1}{z-2})^n = \sum_{n=0}^{\infty}\frac{2(-1)^n}{(z-2)^{n+1}}.$$

2. We know $f(z) = \sin(\sin z) - \sin z$ has a zero at $z_0 = 0$. Find its order.

   Solution: Since $\sin z = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)!}z^{2n+1} = z - \dfrac{z^3}{6} + \dfrac{z^5}{120} - \cdots$ for all z, we have

$$\begin{aligned}\sin(\sin z) &= \sin z - \frac{(\sin z)^3}{6} + \frac{(\sin z)^5}{120} - \cdots \\ &= (z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots) - \frac{(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots)^3}{6} + \frac{(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots)^5}{120} - \cdots \\ &= z - \frac{z^3}{6} - \frac{z^3}{6} + \cdots = z - \frac{z^3}{3} + \cdots.\end{aligned}$$

   It follows that

$$f(z) = \sin(\sin z) - \sin z = (z - \frac{z^3}{3} + \cdots) - (z - \frac{z^3}{6} + \cdots) = -\frac{z^3}{6} + \cdots.$$

   Thus f has a zero of order 3 at $z_0 = 0$.

3. Find the residues of f at all its isolated singular points.

   (a) $f(z) = \dfrac{z^2}{(z^2+1)^2}$.

   Solution: To find singular points, set $(z^2+1)^2 = 0$ and solve it to obtain $z = \pm i$.

   $\mathrm{Res}_{z=i}\dfrac{z^2}{(z^2+1)^2} = \mathrm{Res}_{z=i}\dfrac{\frac{z^2}{(z+i)^2}}{(z-i)^2} = \phi'(i)$, where $\phi(z) = \dfrac{z^2}{(z+i)^2}$.

Since $\phi'(z) = \dfrac{2z(z+i)^2 - z^2(2(z+i))}{(z+i)^4}$, we have

$$\text{Res}_{z=i}\, \frac{z^2}{(z^2+1)^2} = \phi'(i) = \frac{2i(i+i)^2 - i^2(2(i+i))}{(i+i)^4} = -\frac{i}{4}.$$

Similarly, $\text{Res}_{z=-i}\, \dfrac{z^2}{(z^2+1)^2} = \text{Res}_{z=-i}\, \dfrac{\frac{z^2}{(z-i)^2}}{(z+i)^2} = \psi'(-i)$, where $\psi(z) = \dfrac{z^2}{(z-i)^2}$.
Since $\psi'(z) = \dfrac{2z(z-i)^2 - z^2(2(z-i))}{(z-i)^4}$, we have

$$\text{Res}_{z=-i}\, \frac{z^2}{(z^2+1)^2} = \psi'(-i) = \frac{2(-i)(-i-i)^2 - (-i)^2(2(-i-i))}{(-i-i)^4} = \frac{i}{4}.$$

(b) $f(z) = \dfrac{z^n}{(z+3)^n}$, where n is a positive integer.

Solution: The only singular point of f is $z = -3$. To find the residue of f at $z = -3$, denote $\phi(z) = z^n$. Then

$$\text{Res}_{z=-3}\, \frac{z^n}{(z+3)^n} = \frac{\phi^{(n-1)}(-3)}{(n-1)!}.$$

Since

$$\phi^{(n-1)}(z) = n(n-1)\cdots 2 \cdot z = n!z,$$

we have

$$\text{Res}_{z=-3}\, \frac{z^n}{(z+3)^n} = \frac{\phi^{(n-1)}(-3)}{(n-1)!} = \frac{n!(-3)}{(n-1)!} = -3n.$$

(c) $f(z) = \dfrac{\cot z}{z}$.

Solution: Note that $\dfrac{\cot z}{z} = \dfrac{\cos z}{z \sin z}$. To find its singular points, set $z \sin z = 0$, and solve it to obtain $z = n\pi$, $n \in \mathbb{Z}$.

Denote by $p(z) = \cos z$ and $q(z) = z \sin z$. Then $q'(z) = \sin z + z \cos z$. If $n \neq 0$, then $q(n\pi) = 0$ and $q'(n\pi) = \sin(n\pi) + n\pi \cos(n\pi) = n\pi \cos(n\pi) \neq 0$. Thus

$$\text{Res}_{z=n\pi}\, \frac{\cos z}{z \sin z} = \frac{p(n\pi)}{q'(n\pi)} = \frac{\cos(n\pi)}{n\pi \cos(n\pi)} = \frac{1}{n\pi}.$$

To find the residue of f at 0, note that

$$z \sin z = z(z - \frac{z^3}{6} + \cdots) = z^2(1 - \frac{z^2}{6} + \cdots).$$

Thus

$$\text{Res}_{z=0}\, \frac{\cos z}{z \sin z} = \text{Res}_{z=0}\, \frac{\cos z}{z^2(1 - \frac{z^2}{6} + \cdots)} = \text{Res}_{z=0}\, \frac{\frac{\cos z}{1 - \frac{z^2}{6} + \cdots}}{z^2} = \phi'(0)$$

where $\phi(z) = \dfrac{\cos z}{1 - \frac{z^2}{6} + \cdots}$.

Since $\phi'(z) = \dfrac{-\sin z(1 - \frac{z^2}{6} + \cdots) - \cos z(-\frac{z}{3} + \cdots)}{(1 - \frac{z^2}{6} + \cdots)^2}$, we have

$$\mathrm{Res}_{z=0} \frac{\cos z}{z \sin z} = \phi'(0) = \frac{-1 \sin 0 - 0 \cos 0}{1^2} = 0.$$

4. Evaluate the following contour integrals.

   (a) $\displaystyle\oint_{|z|=6} \frac{\cos z}{z^2(z-\pi)^3}\,dz.$

   Solution: The singular points of $\dfrac{\cos z}{z^2(z-\pi)^3}$ inside the circle $|z| = 6$ are 0 and π. we first compute the residues of $\dfrac{\cos z}{z^2(z-\pi)^3}$ at 0 and π.

   $\mathrm{Res}_{z=0} \dfrac{\cos z}{z^2(z-\pi)^3} = \mathrm{Res}_{z=0} \dfrac{\frac{\cos z}{(z-\pi)^3}}{z^2} = \dfrac{d}{dz}\dfrac{\cos z}{(z-\pi)^3}\ |_{z=0}$
   $= \dfrac{-\sin z(z-\pi)^3 - \cos z(3(z-\pi)^2)}{(z-\pi)^6}\ |_{z=0}= \dfrac{-3(0-\pi)^2}{(0-\pi)^6} = -\dfrac{3}{\pi^4}$, and
   $\mathrm{Res}_{z=\pi} \dfrac{\cos z}{z^2(z-\pi)^3} = \mathrm{Res}_{z=\pi} \dfrac{\frac{\cos z}{z^2}}{(z-\pi)^3} = \dfrac{1}{2}\dfrac{d^2}{dz^2}\dfrac{\cos z}{z^2}\ |_{z=\pi}$
   $= \dfrac{1}{2}\dfrac{d}{dz}\dfrac{-z^2 \sin z - 2z \cos z}{z^4}\ |_{z=\pi}= \dfrac{1}{2}\dfrac{d}{dz}\dfrac{-z \sin z - 2 \cos z}{z^3}\ |_{z=\pi}$
   $= \dfrac{1}{2}\dfrac{(-\sin z - z\cos z + 2\sin z)(z^3) - (-z\sin z - 2\cos z)(3z^2)}{z^6}\ |_{z=\pi}$
   $= \dfrac{(-\sin \pi - \pi\cos \pi + 2\sin \pi)(\pi^3) - (-\pi\sin \pi - 2\cos \pi)(3\pi^2)}{2\pi^6}$
   $= \dfrac{\pi^4 - 6\pi^2}{2\pi^6} = \dfrac{\pi^2 - 6}{2\pi^4}$. Thus we have
   $\displaystyle\oint_{|z|=6} \frac{\cos z}{z^2(z-\pi)^3}\,dz = 2\pi i(\mathrm{Res}_{z=0} \frac{\cos z}{z^2(z-\pi)^3} + \mathrm{Res}_{z=\pi} \frac{\cos z}{z^2(z-\pi)^3})$
   $= 2\pi i(-\dfrac{3}{\pi^4} + \dfrac{\pi^2 - 6}{2\pi^4}) = 2\pi i(\dfrac{\pi^2 - 12}{2\pi^4}) = \dfrac{\pi^2 - 12}{\pi^3}i.$

   (b) $\displaystyle\oint_{|z|=10} \frac{z}{e^z - 1}\,dz.$

   Solution: First, to find the singular points of $\dfrac{z}{e^z - 1}$, set $e^z - 1 = 0$ and solve it to obtain $z = 2n\pi i$, $n \in \mathbb{Z}$. So all singular points inside the circle $|z| = 10$ are: $0, \pm 2\pi i$. Next, we compute the residues of $\dfrac{z}{e^z - 1}$ at each singular point.

   $\mathrm{Res}_{z=0} \dfrac{z}{e^z - 1} = \dfrac{z}{\frac{d}{dz}(e^z - 1)}\ |_{z=0}= \dfrac{z}{e^z}\ |_{z=0}= \dfrac{0}{e^0} = 0,$
   $\mathrm{Res}_{z=2\pi i} \dfrac{z}{e^z - 1} = \dfrac{z}{\frac{d}{dz}(e^z - 1)}\ |_{z=2\pi i}= \dfrac{z}{e^z}\ |_{z=2\pi i}= \dfrac{2\pi i}{e^{2\pi i}} = 2\pi i$, and
   $\mathrm{Res}_{z=-2\pi i} \dfrac{z}{e^z - 1} = \dfrac{z}{\frac{d}{dz}(e^z - 1)}\ |_{z=-2\pi i}= \dfrac{z}{e^z}\ |_{z=-2\pi i}= \dfrac{-2\pi i}{e^{-2\pi i}} = -2\pi i$. Thus we have

   $$\oint_{|z|=10} \frac{z}{e^z - 1}\,dz = 2\pi i(0 + 2\pi i + (-2\pi i)) = 0.$$

(c) $\oint_{|z|=1} e^{(\frac{1}{z})} \sin(\frac{1}{z})\, dz$.

Solution: The only singular point of $e^{(\frac{1}{z})} \sin(\frac{1}{z})$ is 0, which is inside the circle $|z| = 1$. To find the residue of $e^{(\frac{1}{z})} \sin(\frac{1}{z})$ at 0, we need to compute its Laurent series in a deleted neighborhood of 0. Note that

$$e^{(\frac{1}{z})} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

and

$$\sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{3!z^3} + \cdots .$$

So we have

$$e^{(\frac{1}{z})} \sin(\frac{1}{z}) = (1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots)(\frac{1}{z} - \frac{1}{3!z^3} + \cdots) = \frac{1}{z} + \frac{1}{z^2} + \cdots .$$

It follows that

$$\mathrm{Res}_{z=0} e^{(\frac{1}{z})} \sin(\frac{1}{z}) = 1.$$

Therefore

$$\oint_{|z|=1} e^{(\frac{1}{z})} \sin(\frac{1}{z})\, dz = 2\pi i.$$

5. Evaluate the following integrals.

(a) $\int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos\theta}\, d\theta$.

Solution: Make the change of variables so the integral will be around the unit circle. $z = e^{i\theta} \Rightarrow \cos\theta = \frac{z + \frac{1}{z}}{2}$, $\sin\theta = \frac{z - \frac{1}{z}}{2i}$, and $d\theta = \frac{dz}{iz}$. So we have

$$\int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos\theta}\, d\theta = \oint_{|z|=1} \frac{(\frac{1}{2i}(z - \frac{1}{z}))^2}{5 + 4(\frac{1}{2}(z + \frac{1}{z}))} \frac{dz}{iz} = \frac{i}{4} \oint_{|z|=1} \frac{z^4 - 2z^2 + 1}{z^2(2z^2 + 5z + 2)}\, dz.$$

Set $z^2(2z^2 + 5z + 2) = 0$ and solve it to obtain $z = 0, -2, -\frac{1}{2}$. So the singular points inside the circle $|z| = 1$ are 0 and $-\frac{1}{2}$. Note that $-\frac{1}{2}$ is a simple pole. We have

$$\mathrm{Res}_{z=-\frac{1}{2}} \frac{z^4 - 2z^2 + 1}{z^2(2z^2 + 5z + 2)} = \frac{z^4 - 2z^2 + 1}{8z^3 + 15z^2 + 4z}\,\Big|_{z=-\frac{1}{2}} = \cdots = \frac{3}{4}.$$

For $z = 0$ which is of order 2, we have

$$\mathrm{Res}_{z=0} \frac{z^4 - 2z^2 + 1}{z^2(2z^2 + 5z + 2)} = \mathrm{Res}_{z=0} \frac{\frac{z^4 - 2z^2 + 1}{2z^2 + 5z + 2}}{z^2} = \frac{d}{dz} \frac{z^4 - 2z^2 + 1}{2z^2 + 5z + 2}\,\Big|_{z=0} = \cdots = -\frac{5}{4}.$$

Therefore

$$\int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos\theta}\, d\theta = \frac{i}{4} \oint_{|z|=1} \frac{z^4 - 2z^2 + 1}{z^2(2z^2 + 5z + 2)}\, dz$$

$$= \frac{i}{4}(2\pi i(\mathrm{Res}_{z=-\frac{1}{2}} \frac{z^4 - 2z^2 + 1}{z^2(2z^2 + 5z + 2)} + \mathrm{Res}_{z=0} \frac{z^4 - 2z^2 + 1}{z^2(2z^2 + 5z + 2)}))$$

$$= \frac{i}{4}(2\pi i)(\frac{3}{4} + \frac{-5}{4}) = \frac{\pi}{4}.$$

(b) $\int_0^\infty \frac{x^2}{(x^2+4)(x^2+9)}\,dx.$

Solution: Let R be a large positive number. Denote by Γ_R the line segment on x-axis from $z=-R$ to $z=R$, by C_R the upper half circle $\{z||z|=R, \text{Im}z \geq 0\}$ from $z=R$ to $z=-R$, and by C the simple closed contour $C=\Gamma_R \bigcup C_R$.

STEP 1: Compute $\int_C \frac{z^2}{(z^2+4)(z^2+9)}\,dz.$

Set $(z^2+4)(z^2+9)=0$ and solve it to obtain $z=\pm 2i$ and $z=\pm 3i$. So the only singular points of $\frac{z^2}{(z^2+4)(z^2+9)}$ inside C are $z=2i$ and $z=3i$. Since both of them are simple poles, we have

$$\text{Res}_{z=2i}\frac{z^2}{(z^2+4)(z^2+9)} = \frac{z^2}{\frac{d}{dz}((z^2+4)(z^2+9))}\,|_{z=2i} = \cdots = \frac{i}{5},$$

and

$$\text{Res}_{z=3i}\frac{z^2}{(z^2+4)(z^2+9)} = \frac{z^2}{\frac{d}{dz}((z^2+4)(z^2+9))}\,|_{z=3i} = \cdots = \frac{-3i}{10}.$$

It follows that

$$\int_C \frac{z^2}{(z^2+4)(z^2+9)}\,dz = 2\pi i(\frac{i}{5}+\frac{-3i}{10}) = \frac{\pi}{5}.$$

STEP 2: Show that $\lim_{R\to\infty}\int_{C_R}\frac{z^2}{(z^2+4)(z^2+9)}\,dz = 0.$

Note that $|z|=R$ when z is on C_R. We have

$$|\frac{z^2}{(z^2+4)(z^2+9)}| = \frac{|z|^2}{|z^2+4||z^2+9|} \leq \frac{|z|^2}{(|z|^2-4)(|z|^2-9)} = \frac{R^2}{(R^2-4)(R^2-9)} = M.$$

Since the length of C_R is equal to $L=\pi R$,

$$|\int_{C_R}\frac{z^2}{(z^2+4)(z^2+9)}\,dz| \leq ML = \frac{R^2}{(R^2-4)(R^2-9)}\pi R = \frac{\pi R^3}{(R^2-4)(R^2-9)}.$$

It follows from simple calculus that $\frac{\pi R^3}{(R^2-4)(R^2-9)} \to 0$ as $R\to\infty$. Therefore

$$\lim_{R\to\infty}\int_{C_R}\frac{z^2}{(z^2+4)(z^2+9)}\,dz = 0.$$

STEP 3: Compute $\int_0^\infty \frac{x^2}{(x^2+4)(x^2+9)}\,dx.$

Since $\frac{x^2}{(x^2+4)(x^2+9)}$ is an even function of x, we have

$$\int_0^\infty \frac{x^2}{(x^2+4)(x^2+9)}\,dx = \frac{1}{2}\lim_{R\to\infty}\int_{\Gamma_R}\frac{z^2}{(z^2+4)(z^2+9)}\,dz$$
$$= \frac{1}{2}(\lim_{R\to\infty}(\int_C \frac{z^2}{(z^2+4)(z^2+9)}\,dz - \int_{C_R}\frac{z^2}{(z^2+4)(z^2+9)}\,dz))$$
$$= \frac{1}{2}(\frac{\pi}{5}-0) = \frac{\pi}{10}.$$

(c) $\displaystyle\int_0^\infty \frac{x\sin x}{(x^2+1)(x^2+4)}\,dx.$

Solution: Let R be a large positive number. Denote by Γ_R the line segment on x-axis from $z=-R$ to $z=R$, by C_R the upper half circle $\{z||z|=R, \mathrm{Im}z \geq 0\}$ from $z=R$ to $z=-R$, and by C the simple closed contour $C=\Gamma_R \bigcup C_R$.

STEP 1: Compute $\displaystyle\int_C \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz.$

Set $(z^2+1)(z^2+4)=0$ and solve it to obtain $z=\pm i$ and $z=\pm 2i$. So the only singular points of $\dfrac{ze^{iz}}{(z^2+1)(z^2+4)}$ inside C are $z=i$ and $z=2i$. Since both of them are simple poles, we have

$$\mathrm{Res}_{z=i}\,\frac{ze^{iz}}{(z^2+1)(z^2+4)} = \frac{ze^{iz}}{\frac{d}{dz}((z^2+1)(z^2+4))}\,|_{z=i} = \cdots = \frac{1}{6e},$$

and

$$\mathrm{Res}_{z=2i}\,\frac{ze^{iz}}{(z^2+1)(z^2+4)} = \frac{ze^{iz}}{\frac{d}{dz}((z^2+1)(z^2+4))}\,|_{z=2i} = \cdots = \frac{-1}{6e^2}.$$

It follows that

$$\int_C \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz = 2\pi i(\frac{1}{6e}+\frac{-1}{6e^2}) = \frac{\pi(e-1)}{3e^2}i.$$

STEP 2: Show that $\displaystyle\lim_{R\to\infty}\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz = 0.$

Note that $z\in C_R \Rightarrow \mathrm{Im}z \geq 0$. It follows that $|e^{iz}|\leq 1$. So we have

$$|\frac{ze^{iz}}{(z^2+1)(z^2+4)}| = \frac{|z||e^{iz}|}{|z^2+1||z^2+4|} \leq \frac{|z|}{(|z|^2-1)(|z|^2-4)} = \frac{R}{(R^2-1)(R^2-4)} = M.$$

Since the length of C_R is equal to $L=\pi R$,

$$|\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz| \leq ML = \frac{R}{(R^2-1)(R^2-4)}\pi R = \frac{\pi R^2}{(R^2-4)(R^2-9)}.$$

It follows from simple calculus that $\dfrac{\pi R^2}{(R^2-1)(R^2-4)} \to 0$ as $R\to\infty$. Therefore

$$\lim_{R\to\infty}\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz = 0.$$

STEP 3: Compute $\displaystyle\int_0^\infty \frac{x\sin x}{(x^2+1)(x^2+4)}\,dx.$

Since $\dfrac{x\sin x}{(x^2+4)(x^2+9)}$ is an even function of x, we have

$\displaystyle\int_0^\infty \frac{x\sin x}{(x^2+4)(x^2+9)}\,dx =$ The imaginary part of $\displaystyle\frac{1}{2}(\lim_{R\to\infty}\int_{\Gamma_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz)$

$=$ The imaginary part of $\displaystyle\frac{1}{2}(\lim_{R\to\infty}(\int_C \frac{z^2}{(z^2+4)(z^2+9)}\,dz - \int_{C_R} \frac{z^2}{(z^2+4)(z^2+9)}\,dz))$

$=$ The imaginary part of $\displaystyle\frac{1}{2}(\frac{\pi(e-1)}{3e^2}i - 0) = \frac{\pi(e-1)}{6e^2}.$

6. Determine the values of $\Delta_C \arg f(z)$ for the function f and positively oriented contour C.

   (a) $f(z) = \dfrac{4z^3+3}{z}$; $C = \{z \mid |z| = 1\}$.

   Solution: The function $f(z) = \dfrac{4z^3+3}{z}$ has 3 zeros $(\dfrac{-3}{4})^{\frac{1}{3}}$, all of which are inside $C = \{z \mid |z| = 1\}$ and of order 1, and it also has one simple pole at 0 inside C. By the Argument Principle, we have

   $$\Delta_C \arg f(z) = 2\pi(Z - P) = 2\pi(3-1) = 4\pi.$$

   (b) $f(z) = \tan z$; $C = \{z \mid |z| = 10\}$.

   Solution: Since $\tan z = \dfrac{\sin z}{\cos z}$, its zeros occur when $\sin z = 0$, and its poles occur when $\cos z = 0$. It follows from simple computations that $f(z) = \tan z$ has 7 zeros $\{0, \pm\pi, \pm 2\pi, \pm 3\pi\}$ inside $C = \{z \mid |z| = 10\}$, all of which are of order 1, and it has 6 simple poles $\{\pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2} \pm \dfrac{5\pi}{2}\}$ inside C. By the Argument Principle, we have

   $$\Delta_C \arg f(z) = 2\pi(Z - P) = 2\pi(7-6) = 2\pi.$$

7. Determine the number of roots, counting multiplicities, of the following equations in the given regions.

   (a) $z^5 + z^2 + 10z + 3 = 0$; $D_1 = \{z \mid |z| < 1\}$, $D_2 = \{z \mid |z| < 2\}$, and $D_3 = \{z \mid 1 \le |z| < 2\}$.
   Solution: 1. $D_1 = \{z \mid |z| < 1\}$.
   Let $f(z) = 10z$, and $g(z) = z^5 + z^2 + 3$. Both are analytic in D_1. When z is on $|z| = 1$,

   $$|f(z)| = 10, \text{ and } |g(z)| = |z^5 + z^2 + 3| \le |z|^5 + |z|^2 + 3 = 5.$$

   It follows that $|f(z)| > |g(z)|$ when z is on $|z| = 1$.
   By Rouché's theorem, $f(z) + g(z) = z^5 + z^2 + 10z + 3 = 0$ has the same number of roots, counting multiplicities, as $f(z) = 10z = 0$ in D_1. Since $10z = 0$ has exactly one root 0 of order 1, we conclude that $z^5 + z^2 + 10z + 3 = 0$ has one root in D_1.

   2. $D_2 = \{z \mid |z| < 2\}$.
   Let $f(z) = z^5$, and $g(z) = z^2 + 10z + 3$. Both are analytic in D_1. When z is on $|z| = 2$,

   $$|f(z)| = |z^5| = |z|^5 = 32, \text{ and } |g(z)| = |z^2 + 10z + 3| \le |z|^2 + 10|z| + 3 = 27.$$

   It follows that $|f(z)| > |g(z)|$ when z is on $|z| = 2$.
   By Rouché's theorem, $f(z) + g(z) = z^5 + z^2 + 10z + 3 = 0$ has the same number of roots, counting multiplicities, as $f(z) = z^5 = 0$ in D_2. Since $z^5 = 0$ has one root $z = 0$ of order 5, we conclude that $z^5 + z^2 + 10z + 3 = 0$ has 5 roots, counting multiplicities, in D_2.

   3. $D_3 = \{z \mid 1 \le |z| < 2\}$.
   It follows from the above that $z^5 + z^2 + 10z + 3 = 0$ has 4 roots, counting multiplicities, in D_3.

(b) $5z^6 = \cos z + 1$; $D = \{z \,|\, |z| < 1\}$.

Solution: Let $f(z) = 5z^6$, and $g(z) = -\cos z - 1$. Both are analytic in D. When z is on $|z| = 1$ we have $|f(z)| = |5z^6| = 5|z|^6 = 5$, and

$$|g(z)| = |-\cos z - 1| = |\frac{e^{iz}}{2} + \frac{e^{-iz}}{2} + 1|| \leq |\frac{e^{iz}}{2}| + |\frac{e^{-iz}}{2}| + 1 \leq \frac{e}{2} + \frac{e}{2} + 1 = e + 1.$$

Here we have used

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{-y+ix}| = e^{-y} \leq e$$

since $-y \leq 1$ on the circle $|z| = 1$; and similarly we have $|e^{-iz}| \leq e$. It follows that $|f(z)| > |g(z)|$ when z is on $|z| = 1$.

By Rouché's theorem, $f(z) + g(z) = 5z^6 - \cos z - 1 = 0$ has the same number of roots, counting multiplicities, as $f(z) = 5z^6 = 0$ in D. Since $5z^6 = 0$ has one root 0 of order 6, we conclude that $5z^6 - \cos z - 1 = 0$, or equivalently, $z^6 = \cos z + 1$ also has 6 roots, counting multiplicities, in D.