Math 421 – Practice Exam II Solutions
April, 2009

- **Instructions for the exam:**

  (1) You need to justify your work.

  (2) Expressions such as $\sin(i\pi)$ in Problem 7 or $-i\log(3i)$ in Problem 4 are not acceptable as the final answer. They need to be simplified.

- **The notation $\oint_C f(z)\,dz$ indicates the simple closed contour C is positively oriented.**

1. True or False. Explain why.

   (a) $\forall z \in \mathbb{C}, 1^z = 1$. **F**

   (b) $\forall z \in \mathbb{C}, z^1 = z$. **T**

   (c) $f(z) = z^c$ is a single-valued function if and only if $c \in \mathbb{Z}$. **T**

   (d) An analytic function f in a domain D must have an antiderivative everywhere in D. **F**

2. Find all value(s) the following expressions.

   (a) $\log(3+3i)$.
   Solution: $\log(3+3i) = \ln|3+3i| + i\arg(3+3i) = \ln(3\sqrt{2}) + i(\frac{\pi}{4} + 2n\pi)$, $n \in \mathbb{Z}$.

   (b) $(ei)^{\pi i}$.
   Solution: $(ei)^{\pi i} = e^{\pi i \log(ei)} = e^{\pi i(\ln|ei| + i\arg(ei))} = e^{\pi i(1+i(\frac{\pi}{2}+2n\pi))} = e^{-(2n+\frac{1}{2})\pi^2+\pi i} = e^{-(2n+\frac{1}{2})\pi^2}(\cos\pi + i\sin\pi) = -e^{-(2n+\frac{1}{2})\pi^2}$, $n \in \mathbb{Z}$.

3. Find the principal value of $(1+i)^i$.

   Solution: The principal value of $(1+i)^i = e^{i\mathrm{Log}(1+i)} = e^{i(\ln|1+i| + i\mathrm{Arg}(1+i))} = e^{i(\ln\sqrt{2}+i\frac{\pi}{4})} = e^{-\frac{\pi}{4}}(\cos(\ln\sqrt{2}) + i\sin(\ln\sqrt{2}))$.

4. Solve the equation $\sin z = \dfrac{5}{3}$ for z.

   Solution: Since $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, we have $\dfrac{e^{iz} - e^{-iz}}{2i} = \dfrac{5}{3}$, so $3(e^{iz} - e^{-iz}) = 10i$. Let $u = e^{iz}$. The equation simplifies to $3(u - \frac{1}{u}) - 10i = 0$. Multiplying by y, we have $3u^2 - 10iu - 3 = 0$. Using quadratic formula, one gets

$$u = \frac{10i + ((-10i)^2 - 4(3)(-3))^{\frac{1}{2}}}{2(3)} = \frac{10i + (-64)^{\frac{1}{2}}}{6} = \frac{10i \pm 8i}{6}.$$

   So $u = 3i$ or $u = \frac{i}{3}$. That is, $e^{iz} = 3i$ or $e^{iz} = \frac{i}{3}$.
   If $e^{iz} = 3i$, $iz = \log(3i)$, then

$$z = -i(\ln 3 + i\arg(3i)) = -i(\ln 3 + i(\frac{\pi}{2} + 2n\pi)) = (2n + \frac{1}{2})\pi - i\ln 3,\ n \in \mathbb{Z}.$$

If $e^{iz} = \frac{i}{3}$, $iz = \log(\frac{i}{3})$, then

$$z = -i(\ln\frac{1}{3} + i\arg(\frac{i}{3})) = -i(\ln\frac{1}{3} + i(\frac{\pi}{2} + 2n\pi)) = (2n + \frac{1}{2})\pi + i\ln 3,\ n \in \mathbb{Z}.$$

5. Show that $|\oint_{|z|=3} \frac{z}{z^4 + 9z^2 + 18}\,dz| \le \pi$.

   Proof: If $|z| = 3$, $|z^4+9z^2+18| = |(z^2+6)||(z^2+3)| \ge (|z|^2-6)(|z|^2-3) = (3^2-6)(3^2-3) = 18$. Thus

   $$|\frac{z}{z^4 + 9z^2 + 18}| = \frac{|z|}{|z^4 + 9z^2 + 18|} \le \frac{3}{18} = \frac{1}{6} = M.$$

   Since the length of the contour $L = 2\pi(3) = 6\pi$, we have

   $$|\oint_{|z|=3} \frac{z}{z^4 + 9z^2 + 18}\,dz| \le ML = \frac{1}{6} \cdot 6\pi = \pi.$$

6. Compute $\int_C \text{Re}z\,dz$, where C is the line segment from 0 to $1 + 2i$.

   Solution: Parametrize the curve C by $z(t) = t + i(2t)$, $0 \le t \le 1$. Since $\text{Re}z = t$ and $z'(t) = 1 + 2i$, we have

   $$\int_C \text{Re}z\,dz = \int_0^1 t \cdot (1 + 2i)\,dt = (1 + 2i)\frac{t^2}{2}|_0^1 = \frac{1}{2} + i.$$

7. Evaluate $\int_C \cos z\,dz$, where C starts at the origin, traverses the bottom half of a unit circle centered at $z_0 = 1$ and then the line segment from to $z = 2$ to $z = i\pi$.

   Solution: Since the function $\cos z$ has an anti-derivative $\sin z$ which is entire, the contour integral $\int_C \cos z\,dz$ is independent of path, and

   $$\int_C \cos z\,dz = \int_0^{i\pi} \cos z dz = \sin z|_0^{i\pi} = \sin(i\pi) = \frac{e^{i(i\pi)} - e^{-i(i\pi)}}{2i} = \frac{e^{\pi} - e^{-\pi}}{2}i.$$

8. Compute $\oint_{|z|=2} \frac{\sin z}{z}\,dz$.

   Solution: Note that $f(z) = \sin z$ is analytic inside and on the circle $|z| = 2$, and $z_0 = 0$ is interior to $|z| = 2$. Thus

   $$\oint_{|z|=2} \frac{\sin z}{z}\,dz = 2\pi i f(0) = 2\pi i \sin 0 = 0.$$

9. Compute $\oint_{|z|=2} \frac{1}{z(z+1)^2(z+3)}\,dz$.

   Solution: Note that the function $\frac{1}{z(z+1)^2(z+3)}$ is analytic everywhere inside and on the circle $|z| = 2$ except at $z = 0$ and $z = -1$. Let C_1 and C_2 be two small circles (of radius 0.1 for instance) centered at $z = 0$ and $z = -1$, respectively. We have

   $$\oint_{|z|=2} \frac{1}{z(z+1)^2(z+3)}\,dz = \oint_{C_1} \frac{1}{z(z+1)^2(z+3)}\,dz + \oint_{C_2} \frac{1}{z(z+1)^2(z+3)}\,dz.$$

To compute $\oint_{C_1} \frac{1}{z(z+1)^2(z+3)}\,dz$, note that $f(z) = \frac{1}{(z+1)^2(z+3)}$ is analytic inside and on the circle C_1. Thus

$$\oint_{C_1} \frac{1}{z(z+1)^2(z+3)}\,dz = \oint_{C_1} \frac{\frac{1}{(z+1)^2(z+3)}}{z}\,dz = 2\pi i f(0) = 2\pi i(\frac{1}{3}) = \frac{2\pi i}{3}.$$

To compute $\oint_{C_2} \frac{1}{z(z+1)^2(z+3)}\,dz$, note that $g(z) = \frac{1}{z(z+3)}$ is analytic inside and on the circle C_2. Thus $g'(z) = \frac{-(2z+3)}{z^2(z+3)^2}$, and

$$\oint_{C_2} \frac{1}{z(z+1)^2(z+3)}\,dz = \oint_{C_1} \frac{\frac{1}{z(z+3)}}{(z+1)^2}\,dz = 2\pi i g'(-1) = 2\pi i(\frac{-(2(-1)+3)}{(-1)^2((-1)+3)^2}) = -\frac{\pi i}{2}.$$

Therefore

$$\oint_{|z|=2} \frac{1}{z(z+1)^2(z+3)}\,dz = \frac{2\pi i}{3} + (-\frac{\pi i}{2}) = \frac{\pi i}{6}.$$

10. Compute $\displaystyle\int_C \frac{z(z^2+9)}{(z^2+1)(z^6+z^2+100)}\,dz$, where C is the upper half circle $|z|=2$ from $z=2$ to $z=-2$.

Solution: Let C_1 denote the line segment on the x-axis from $z=-2$ to $z=2$. So $C_2 = C\bigcup C_1$ is a simple closed contour, and if we write $f(z) = \dfrac{z(z^2+9)}{(z^2+1)(z^6+z^2+100)}$, we have

$$\int_{C_2} f(z)\,dz = \int_C f(z)\,dz + \int_{C_1} f(z)dz.$$

Thus

$$\int_C f(z)\,dz = \int_{C_2} f(z)\,dz - \int_{C_1} f(z)\,dz.$$

To compute $\displaystyle\int_{C_2} f(z)\,dz$ where C_2 is a closed contour, we first identify all points interior to C_2 where $f(z)$ fails to be analytic, i.e. where the denominator of $f(z)$ vanishes. If $z^2+1=0$, then $z=\pm i$. Since $z=-i$ is outside C_2, it should not be considered. To show that $z=i$ is the only point interior to C_2 where $f(z)$ is not analytic, it suffices to show that $z^6+z^2+100 \neq 0$ whenever $|z|<2$. This is true because if $|z|<2$, then

$$|z^6+z^2+100| \geq 100 - |z^6| - |z^2| \geq 100 - 2^6 - 2^2 = 32 > 0.$$

Thus we have

$$\int_{C_2} \frac{z(z^2+9)}{(z^2+1)(z^6+z^2+100)}\,dz = \int_{C_2} \frac{\frac{z(z^2+9)}{(z+i)(z^6+z^2+100)}}{z-i}\,dz = 2\pi i\frac{i(i^2+9)}{(i+i)(i^6+i^2+100)} = \frac{4\pi i}{49}.$$

Now we compute $\displaystyle\int_{C_1} \frac{z(z^2+9)}{(z^2+1)(z^6+z^2+100)}\,dz$. Since C_1 is the line segment on the x-axis from $z=-2$ to $z=2$, we have

$$\int_{C_1} \frac{z(z^2+9)}{(z^2+1)(z^6+z^2+100)}\,dz = \int_{-2}^{2} \frac{x(x^2+9)}{(x^2+1)(x^6+x^2+100)}\,dx = 0.$$

The second equality holds since $\dfrac{x(x^2+9)}{(x^2+1)(x^6+x^2+100)}$ is an odd function of x.

Therefore

$$\int_C \frac{z(z^2+9)}{(z^2+1)(z^6+z^2+100)}\,dz = \frac{4\pi i}{49} - 0 = \frac{4\pi i}{49}.$$

11. Compute $\displaystyle\oint_{|z|=1} \frac{e^{z^2}}{z}\,dz$. Then use it to evaluate $\displaystyle\int_0^\pi e^{\cos 2\theta}\cos(\sin 2\theta)\,d\theta$.

Solution: Note that $f(z) = e^{z^2}$ is analytic inside and on the circle $|z| = 1$, and $z_0 = 0$ is interior to $|z| = 1$. Thus

$$\oint_{|z|=1} \frac{e^{z^2}}{z}\,dz = 2\pi i f(0) = 2\pi i e^0 = 2\pi i.$$

Now we parametrize the circle $|z| = 1$ as $z(\theta) = e^{i\theta}$, $-\pi \le \theta \le \pi$. Then $z'(\theta) = ie^{i\theta}$, and

$$\oint_{|z|=1} \frac{e^{z^2}}{z}\,dz = \int_{-\pi}^{\pi} \frac{e^{e^{2i\theta}}}{e^{i\theta}} ie^{i\theta} d\theta = i\int_{-\pi}^{\pi} e^{\cos 2\theta + i\sin 2\theta} d\theta = i\int_{-\pi}^{\pi} e^{\cos 2\theta}(\cos(\sin 2\theta) + i\sin(\sin 2\theta))d\theta.$$

Since we have already known that $\oint_{|z|=1} \frac{e^{z^2}}{z}\,dz = 2\pi i$, we have

$$\int_{-\pi}^{\pi} e^{\cos 2\theta}(\cos(\sin 2\theta) + i\sin(\sin 2\theta))d\theta = 2\pi.$$

Compare the real parts of both sides, we have

$$\int_{-\pi}^{\pi} e^{\cos 2\theta}\cos(\sin 2\theta)d\theta = 2\pi.$$

Note that the integrand $e^{\cos 2\theta}(\cos(\sin 2\theta)$ is an even function of θ. We conclude that

$$\int_0^{\pi} e^{\cos 2\theta}\cos(\sin 2\theta)d\theta = \pi.$$

12. Show that the area of a region enclosed by a simple closed contour C is equal to $\displaystyle\frac{1}{2i}\oint_C \bar{z}\,dz$.

Proof: Since $z = x + iy$, $\bar{z} = x - iy$ and $dz = dx + idy$. Thus

$$\frac{1}{2i}\oint_C \bar{z}dz = \frac{1}{2i}\oint_C (x - iy)(dx + idy) = \frac{1}{2i}(\oint_C xdx + ydy + i\oint_C -ydx + xdy).$$

Using Green's theorem

$$\oint_C P(x,y)dx + Q(x,y)dy = \int\int_R Q_x - P_y dA,$$

where R is the region enclosed by the curve C, we have

$$\oint_C xdx + ydy = \int\int_R 0 - 0dA = 0,$$

and

$$\oint_C -ydx + xdy = \int\int_R 1 - (-1)dA = 2\int\int_R 1dA = 2 \cdot \text{Area}(R).$$

Therefore

$$\frac{1}{2i}\oint_C \bar{z}dz = \frac{1}{2i}(0 + i(2 \cdot \text{Area}(R))) = \text{Area}(R).$$

13. Let f be an entire function. Show that if there exist positive A and B such that $|f(z)| \leq A|z|^{\frac{1}{2}} + B$ for all z, then f is constant.

 Proof: It suffices to show that $f'(z) = 0$ for all z. Given any z_0, denote by C_R the circle $|z - z_0| = R$. If z is on C_R, we have $|z| = |(z - z_0) + z_0| \leq |z - z_0| + |z_0| = R + |z_0|$. Thus

$$|f(z)| \leq A|z|^{\frac{1}{2}} + B \leq A(R + |z_0|)^{\frac{1}{2}} + B = M_R.$$

 Thus we have

$$|f'(z_0)| \leq \frac{1!M_R}{R^1} = \frac{A(R + |z_0|)^{\frac{1}{2}} + B}{R}.$$

 Since the limit of the right side, as R approached ∞, is equal to 0, we have $|f'(z_0)| \leq 0$. This implies $|f'(z_0)| = 0$, hence $f'(z_0) = 0$ for all z_0. It follows that f is constant.

14. Let f be an entire function. Show that if there exist positive A, B, and C such that $|f(z)| \leq A|z|^2 + B|z| + C$ for all z, then f is a polynomial of degree at most 2. Can you generalize this?

 Proof: It suffices to show that $f'''(z) = 0$ for all z. Given any z_0, denote by C_R the circle $|z - z_0| = R$. If z is on C_R, we have $|z| = |(z - z_0) + z_0| \leq |z - z_0| + |z_0| = R + |z_0|$. Thus

$$|f(z)| \leq A|z|^2 + B|z| + C \leq A(R + |z_0|)^2 + B(R + |z_0|) + C = M_R.$$

 Thus we have

$$|f'''(z_0)| \leq \frac{3!M_R}{R^3} = \frac{6(A(R + |z_0|)^2 + B(R + |z_0|) + C)}{R^3}.$$

 Since the limit of the right side, as R approached ∞, is equal to 0, we have $|f'''(z_0)| \leq 0$. This implies $|f'''(z_0)| = 0$, hence $f'(z_0) = 0$ for all z_0. It follows that f is a polynomial of degree at most 2.

 The generalization: Let f be an entire function. Suppose there exist positive $A_0, A_1, ..., A_n$ such that $|f(z)| \leq A_0 + A_1|z| + ...A_n|z|^n$ for all z, then f is a polynomial of degree at most n.

15. Define $g(z) = \oint_{|s|=2} \frac{s^3 - 6s^2 + 12s + 5}{(s - z)^2} ds$ for all z such that $|z| \neq 2$.

 (a) Compute $g(i)$ and $g(3 + 4i)$.

 Solution: Let $f(z) = z^3 - 6z^2 + 12z + 5$. Then $f'(z) = 3z^2 - 12z + 12$.

 Since f is analytic inside and on the circle $|s| = 2$ and $z = i$ is interior to $|s| = 2$. Then

$$g(i) = \oint_{|s|=2} \frac{s^3 - 6s^2 + 12s + 5}{(s - i)^2} ds = 2\pi i f'(i) = 2\pi i(3i^2 - 12i + 12) = 24\pi + 18\pi i.$$

Since $z = 3 + 4i$ is outside the circle $|s| = 2$, the function $\dfrac{s^3 - 6s^2 + 12s + 5}{(s - (3 + 4i))^2}$ is analytic inside and on the circle $|s| = 2$. Thus

$$g(3 + 4i) = \oint_{|s|=2} \frac{s^3 - 6s^2 + 12s + 5}{(s - (3 + 4i))^2}\, ds = 0.$$

(b) Find all value(s) of z such that $|z| \neq 2$ and $g(z) = 6\pi i$.

Solution: If z is inside the circle $|s| = 2$, then

$$g(z) = 2\pi i f'(z) = 2\pi i(3z^2 - 12z + 12).$$

If z is outside the circle $|s| = 2$, then $g(z) = 0$. So $g(z) = 6\pi i$ can only occur when z is inside the circle $|s| = 2$. To find such values of z, set

$$g(z) = 2\pi i(3z^2 - 12z + 12) = 6\pi i.$$

One gets $z = 1$ or $z = 3$. It follows that the only solutions to $g(z) = 6\pi i$ is $z = 1$ since $z = 3$ is not inside the circle $|s| = 2$.

(c) Find all value(s) of z such that $|z| \neq 2$ and $g(z) = 0$.

Solution: We already know that $g(z) = 0$ for all z outside the circle $|s| = 2$. That is, all values of z such that $|z| > 2$ are solutions to $g(z) = 0$.

To find the values of z inside the circle $|s| = 2$ such that $g(z) = 0$, set

$$g(z) = 2\pi i(3z^2 - 12z + 12) = 0.$$

One gets $z = 2$. But $z = 2$ is not inside $|s| = 2$. It follows that the only solutions to $g(z) = 0$ are all z's such that $|z| > 2$.