Math 421 – Practice Exam II
April, 2009

The notation $\oint_C f(z)\,dz$ indicates the simple closed contour C is positively oriented.

1. True or False. Explain why.

   (a) $\forall z \in \mathbb{C}, 1^z = 1$.

   (b) $\forall z \in \mathbb{C}, z^1 = z$.

   (c) $f(z) = z^c$ is a single-valued function if and only if $c \in \mathbb{Z}$.

   (d) An analytic function f in a domain D must have an antiderivative everywhere in D.

2. Find all value(s) the following expressions.

   (a) $\log(3 + 3i)$.

   (b) $(ei)^{\pi i}$.

3. Find the principal value of $(1 + i)^i$.

4. Solve the equation $\sin z = \dfrac{5}{3}$ for z.

5. Show that $|\displaystyle\oint_{|z|=3} \frac{z}{z^4 + 9z^2 + 18}\,dz| \le \pi$.

6. Compute $\displaystyle\int_C \mathrm{Re}\,z\,dz$, where C is the line segment from 0 to $1 + 2i$.

7. Evaluate $\displaystyle\int_C \cos z\,dz$, where C starts at the origin, traverses the bottom half of a unit circle centered at $z_0 = 1$ and then the line segment from to $z = 2$ to $z = i\pi$.

8. Compute $\displaystyle\oint_{|z|=2} \frac{\sin z}{z}\,dz$.

9. Compute $\displaystyle\oint_{|z|=2} \frac{1}{z(z+1)^2(z+3)}\,dz$.

10. Compute $\displaystyle\int_C \frac{z(z^2+9)}{(z^2+1)(z^6+z^2+100)}\,dz$, where C is the upper half circle $|z| = 2$ from $z = 2$ to $z = -2$.

11. Compute $\displaystyle\oint_{|z|=1} \frac{e^{z^2}}{z}\,dz$. Then use it to evaluate $\displaystyle\int_0^\pi e^{\cos 2\theta}\cos(\sin 2\theta)\,d\theta$.

12. Show that the area of a region enclosed by a simple closed contour C is equal to $\dfrac{1}{2i}\displaystyle\oint_C \bar{z}\,dz$.

13. Let f be an entire function. Show that if there exist positive A and B such that $|f(z)| \le A|z|^{\frac{1}{2}} + B$ for all z, then f is constant.

14. Let f be an entire function. Show that if there exist positive A, B, and C such that $|f(z)| \leq A|z|^2 + B|z| + C$ for all z, then f is a polynomial of degree at most 2. Can you generalize this?

15. Define $g(z) = \oint_{|s|=2} \dfrac{s^3 - 6s^2 + 12s + 5}{(s-z)^2}\, ds$ for all z such that $|z| \neq 2$.

    (a) Compute $g(i)$ and $g(3+4i)$.

    (b) Find all value(s) of z such that $|z| \neq 2$ and $g(z) = 6\pi i$.

    (c) Find all value(s) of z such that $|z| \neq 2$ and $g(z) = 0$.