Math 421 – Practice Exam I Solutions
February, 2009

1. Show that if $|z| = 3$, then $|z^4 + 9z^2 + 20| \geq 20$.

   Solution: $|z^4 + 9z^2 + 20| = |(z^2 + 4)(z^2 + 5)| \geq (|z^2| - 4)(|z^2| - 5) = 5 \cdot 4 = 20$

2. Sketch the set of points determined by $|(2\bar{z} + 6i)(\sqrt{3} + i)| = 8$.

   Solution: $|(2\bar{z} + 6i)(\sqrt{3} + i)| = 8 \Rightarrow |\bar{z} + 3i||2||\sqrt{3} + i| = 8 \Rightarrow |\bar{z} + 3i| = 2 \Rightarrow |\overline{\bar{z} + 3i}| = 2 \Rightarrow$ $|\bar{\bar{z}} + \overline{3i}| = 2 \Rightarrow |z - 3i| = 2$. Thus it's a circle with center $3i$ and radius 2.

3. Solve the equation $(z - 2)^4 + 4 = 0$.

   Solution: $z - 2 \in (-4)^{\frac{1}{4}} = \{1 + i, -1 + i, -1 - i, 1 - i\} \Rightarrow z \in \{3 + i, 1 + i, 1 - i, 3 - i\}$.

4. The four roots $(\sqrt{15} + i)^{\frac{1}{4}}$ of $\sqrt{15} + i$ are vertices of a square. Find the area of the square without actually computing these roots.

   Solution: $|\sqrt{15} + i| = 4 \Rightarrow |(\sqrt{15} + i)^{\frac{1}{4}}| = \sqrt{2}$. Since each side of the square has length 2, the area of the square is 4.

5. Let $f(z) = (y - 1)^3 + i(x - 2)^3$, $z = x + iy$. Show that f is only differentiable when $z = 2 + i$, and find $f'(2 + i)$.

   Solution: $u(x, y) = (y - 1)^3$ and $v(x, y) = (x - 2)^3 \Rightarrow u_x = 0, v_y = 0; u_y = 3(y - 1)^2, v_x = 3(x - 2)^2$. If f is differentiable, the C-R equations $u_x = v_y$ and $u_y = -v_x$ must be satisfied. That is, $0 = 0$ and $3(y - 1)^2 = -3(x - 2)^2$. Thus $x = 2, y = 1$.

   Now at $(2, 1)$, since u_x, u_y, v_x, v_y all exist in some neighborhood of $(2, 1)$ and are all continuous at $(2, 1)$, f is in deed differentiable. And $f'(2 + i) = u_x(2, 1) + iv_x(2, 1) = 0 + i3(2 - 2)^2 = 0$.

6. Show that $g(z) = e^{-x^2+y^2} \cos(2xy) - ie^{-x^2+y^2} \sin(2xy)$, $z = x + iy$ is an entire function.

   Solution: Since

   $$u(x, y) = e^{-x^2+y^2} \cos(2xy), \text{ and } v(x, y) = -e^{-x^2+y^2} \sin(2xy),$$

   we have

   $$u_x = -2xe^{-x^2+y^2} \cos(2xy) - 2ye^{-x^2+y^2} \sin(2xy), v_y = -2xe^{-x^2+y^2} \cos(2xy) - 2ye^{-x^2+y^2} \sin(2xy),$$

   and

   $$u_y = 2ye^{-x^2+y^2} \cos(2xy) - 2xe^{-x^2+y^2} \sin(2xy), v_x = -2ye^{-x^2+y^2} \cos(2xy) + 2xe^{-x^2+y^2} \sin(2xy).$$

   Thus the C-R equations $u_x = v_y$ and $u_y = -v_x$ are satisfied. Since u_x, u_y, v_x, v_y all exist and are all continuous everywhere, we conclude that f is an entire function.

7. (a) Use ϵ-δ definition to show that $\lim_{z \to 0} \frac{\bar{z}^3}{z^2} = 0$.

   Solution: $\forall \epsilon > 0$, pick $\delta = \epsilon$. If $0 < |z - 0| < \delta = \epsilon$, we have $|\frac{\bar{z}^3}{z^2} - 0| = \frac{|\bar{z}|^3}{|z|^2} = |z| < \epsilon$. Thus by definition $\lim_{z \to 0} \frac{\bar{z}^3}{z^2} = 0$.

(b) Show that $\lim_{z\to 0} \dfrac{\bar{z}^3}{z^3}$ does not exist.

Solution: When $z = x$, $\lim_{x\to 0} \dfrac{\bar{x}^3}{x^3} = \lim_{x\to 0} \dfrac{x^3}{x^3} = 1$.

When $z = iy$, $\lim_{y\to 0} \dfrac{(\overline{iy})^3}{(iy)^3} = \lim_{y\to 0} \dfrac{(-iy)^3}{(iy)^3} = -1$.

Since $1 \neq -1$, $\lim_{z\to 0} \dfrac{\bar{z}^3}{z^3}$ does not exist.

(c) Let $f(z) = \begin{cases} \frac{\bar{z}^3}{z^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$ Show that $f(z)$ is not differentiable when $z = 0$.

Solution: $f'(0) = \lim_{z\to 0} \dfrac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \dfrac{\frac{\bar{z}^3}{z^2} - 0}{z - 0} = \lim_{z\to 0} \dfrac{\bar{z}^3}{z^3}$ does not exist by part (a).

(d) Let $g(z) = \begin{cases} \frac{\bar{z}^3}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$ Show that $g(z)$ is differentiable when $z = 0$, and $g'(0) = 0$.

Solution: $g'(0) = \lim_{z\to 0} \dfrac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \dfrac{\frac{\bar{z}^3}{z} - 0}{z - 0} = \lim_{z\to 0} \dfrac{\bar{z}^3}{z^2} = 0$ by part (b).

8. Let a function f be analytic everywhere in a domain D. Suppose that the principal argument $Arg\, f(z) = \dfrac{\pi}{4}$ for all $z \in D$. Prove that $f(z)$ must be constant throughout D.

   Solution: Let $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$. The assumption $Arg\, f(z) = \dfrac{\pi}{4} \Rightarrow$ $v(x,y) = u(x,y)$, so $f(z) = u(x,y) + iu(x,y)$. Since f be analytic in D, the C-R equations must be satisfied. That is $u_x = u_y$ and $u_y = -u_x$. Thus $u_x = u_y = 0$, which implies $u(x,y)$, hence $v(x,y)$ is constant throughout D. Therefore $f(z)$ must be constant throughout D as well.

9. Suppose $f(z) = u(x,y) + iv(x,y)$, $x + iy$ is an entire function and $u(x,y) = -x^3 + 3xy^2 + 2x$. Find $v(x,y)$.

   Solution: By C-R equations, we have $v_x = -u_y = -6xy$ and $v_y = u_x = -3x^2 + 3y^2 + 2$. Thus $v(x,y) = -3x^2y + f(y)$ and $v(x,y) = -3x^2y + y^3 + 2y + g(x)$. Comparing the the two expressions of $v(x,y)$, we get $v(x,y) = -3x^2y + y^3 + 2y + C$.

10. Show that if $f(z) = u(x,y) + iv(x,y)$, $x + iy$ and $g(z) = v(x,y) + iu(x,y)$, $x + iy$ are both analytic in a domain D, then both $f(z)$ and $g(z)$ must be constant throughout D.

    Solution: $f(z) = u(x,y) + iv(x,y)$ is analytic implies $u_x = v_y$ and $u_y = -v_x$; and $g(z) = v(x,y) + iu(x,y)$ is analytic implies $v_x = u_y$ and $v_y = -u_x$. From these equations, one gets $u_x = u_y = v_x = v_y = 0$. Thus $u(x,y)$ and $v(x,y)$ are both constants, hence $f(z)$ and $g(z)$ must be constant throughout D as well.