Math 421 – Practice Exam I
February, 2009

1. Show that if $|z| = 3$, then $|z^4 + 9z^2 + 20| \geq 20$.

2. Sketch the set of points determined by $|(2\bar{z} + 6i)(\sqrt{3} + i)| = 8$.

3. Solve the equation $(z - 2)^4 + 4 = 0$.

4. The four roots $(\sqrt{15} + i)^{\frac{1}{4}}$ of $\sqrt{15} + i$ are vertices of a square. Find the area of the square without actually computing these roots.

5. Let $f(z) = (y - 1)^3 + i(x - 2)^3$, $z = x + iy$. Show that f is only differentiable when $z = 2 + i$, and find $f'(2 + i)$.

6. Show that $g(z) = e^{-x^2+y^2}\cos(2xy) - ie^{-x^2+y^2}\sin(2xy)$, $z = x + iy$ is an entire function.

7. (a) Use ϵ-δ definition to show that $\lim_{z\to 0} \frac{\bar{z}^3}{z^2} = 0$.

   (b) Show that $\lim_{z\to 0} \frac{\bar{z}^3}{z^3}$ does not exist.

   (c) Let $f(z) = \begin{cases} \frac{\bar{z}^3}{z^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$ Show that $f(z)$ is not differentiable when $z = 0$.

   (d) Let $g(z) = \begin{cases} \frac{\bar{z}^3}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$ Show that $g(z)$ is differentiable when $z = 0$, and $g'(0) = 0$.

8. Let a function f be analytic everywhere in a domain D. Suppose that the principal argument $Arg\, f(z) = \frac{\pi}{4}$ for all $z \in D$. Prove that $f(z)$ must be constant throughout D.

9. Suppose $f(z) = u(x, y) + iv(x, y)$, $x + iy$ is an entire function and $u(x, y) = -x^3 + 3xy^2 + 2x$. Find $v(x, y)$.

10. Show that if $f(z) = u(x, y) + iv(x, y)$, $x + iy$ and $g(z) = v(x, y) + iu(x, y)$, $x + iy$ are both analytic in a domain D, then both $f(z)$ and $g(z)$ must be constant throughout D.