Math 300 Spring 2008 - Note 5

Zhigang Han, Umass at Amherst

1 Rational and Real Numbers

1.1 Rational Numbers

Def 1: Define the equivalence relation $\sim$ on the set

$$\mathbb{Z} \times \mathbb{Z} \setminus \{0\} \;=\; \{(a, b) \,|\, a, b \in \mathbb{Z},\; b \neq 0\}$$

by

$$(a, b) \sim (c, d) \;\text{ if and only if }\; ad = bc.$$

The equivalence classes are called **rational numbers**, and the equivalence class containing (a, b) is denoted by $\frac{a}{b}$. The set of all rational numbers is denoted by $\mathbb{Q}$. That is,

$$\mathbb{Q} \;=\; \{\frac{a}{b} \,|\, a, b \in \mathbb{Z},\; b \neq 0\}.$$

Eg: Show that the relation $\sim$ above is an equivalence relation.

Def 2: Define the addition $+$ and the multiplication $\cdot$ on $\mathbb{Q}$ by

$$\frac{a}{b} + \frac{c}{d} \;=\; \frac{ad + bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} \;=\; \frac{ac}{bd}.$$

Eg: Show that these operations are well defined. That is, the definitions are independent of the choice of representatives of the equivalence class.

Prop 1: Any nonzero rational number q can be expressed in a unique way as $q = \frac{a}{b}$ with $b > 0$ and $\gcd(a, b) = 1$.

Rmk: We say the fraction is reduced to its **lowest terms** in this case.

1.2 Real Numbers

Def 1: Let A be the set of infinite sequences $a = (a_1, a_2, \cdots, a_n, \cdots)$ which converges and where $a_i \in \mathbb{Q}$ for all $i \geq 1$. Define the equivalence relation $\sim$ on the set A by

$$a \sim b \text{ if and only if } \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

The equivalence classes are called **real numbers**. The set of all real numbers is denoted by $\mathbb{R}$.

Eg: Show that the relation $\sim$ above is an equivalence relation.

Rmk: Essentially the above definition means that a real number can be thought of as the limit of an infinite sequence of rational numbers. A familiar interpretation of real numbers as infinite decimals uses the same idea.

Rmk: A real number which is not rational is said to be **irrational**. The first known example of irrational numbers is $\sqrt{2}$. Each real number corresponds a unique point on the **number line**. Although rational numbers seem to be **"everywhere"** (what does it mean?) on the number line, it turns out that there are **"many more"** (what does this mean?) irrational numbers than rational numbers.

Def 2: Define the addition $+$ and the multiplication $\cdot$ on $\mathbb{R}$ by

$$[a] + [b] = [a + b]$$

$$[a] \cdot [b] = [a \cdot b].$$

Here for $a = (a_1, a_2, \cdots, a_n, \cdots)$ and $b = (b_1, b_2, \cdots, b_n, \cdots)$, we have

$$a + b = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n, \cdots)$$

and

$$a \cdot b = (a_1 \cdot b_1, a_2 \cdot b_2, \cdots, a_n \cdot b_n, \cdots).$$

Eg: Show that these operations are well defined. That is, the definitions are independent of the choice of representatives of the equivalence class.

1.3 Decimal Expansions

Def 3: The expression $b.a_1a_2\cdots$, where $b \in \mathbb{Z}$ and each a_i is a digit from 0 to 9, is called the decimal expansion of the real number r if

$$b.a_1a_2\cdots a_n \leq r \leq 10^{-n} + b.a_1a_2\cdots a_n \text{ for all } n \in \mathbb{N}.$$

Rmk: Certain numbers have two different decimal expansions. For instance, $\frac{1}{8} = 0.125000\cdots = 0.124999\cdots$.

Def 4: A decimal $b.a_1a_2\cdots$ is **terminating** if there exists an integer n such that $a_i = 0$ for all $i \geq n$. A decimal is called **periodic** if there exists positive integers p and n such that $a_{i+p} = a_i$ for all $i \geq n$.

Eg 1: $\frac{1}{4} = 0.25000\cdots = 0.25$ is terminating. $\frac{1}{7} = 0.\dot{1}\dot{4}\dot{2}\dot{8}\dot{5}\dot{7}$ is periodic.
Eg 2: Find the rational number with the decimal expansion $1.2\dot{3}\dot{4}$.

Theorem: A real number is rational if and only if its decimal expansion is periodic or terminating. Hence a real number is irrational if and only if it has a nonperiodic infinite decimal expansion.