Math 300.01 – Spring 2008

Practice Exam

1. (12 pts) Let G be a group and $g \in G$ is a fixed element. Define a function $f : G \to G$ by $f(x) = g^{-1}xg$. Show that f is a group isomorphism.

2. (12 pts) Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Suppose $g \circ f : X \to Z$ is injective.

   (a) (6 pts) Show that $f : X \to Y$ is also injective.

   (b) (6 pts) Give a counter-example to show that $g : Y \to Z$ is not necessarily injective.

3. (12 pts) Show that $X = \{x \,|\, 0 < x < 1\}$ and $Y = \{y \,|\, y > 0\}$ have the same cardinality by constructing an explicit bijection $f : X \to Y$.

4. (12 pts) Let $f : (\mathbb{Z}, +) \to (\mathbb{Z}_5, +)$ be defined by $f(n) = [n]$.

   (a) (6 pts) Show that f is a group homomorphism.

   (b) (6 pts) Find the Kernel of f.

5. (12 pts) Let $f : G_1 \to G_2$ be a group homomorphism, and H be a subgroup of G_1. Define
$$K = \{x \,|\, x \in H \text{ and } f(x) = e_2\}$$
where e_2 is the identity element in G_2. Show that K is a subgroup of G_1.

6. (**Bonus** 10 pts) Let G be a group with finite many elements and H be a subgroup of G. Denote by $\#G$ and $\#H$ the number of elements in G and in H respectively. Prove that $\#H$ divides $\#G$.

   Hint: For each $a \in G$, define the coset $aH = \{ah \in G \,|\, h \in H\}$.

   (i) Show that there is a bijection from H to aH.

   (ii) Show that for $a, b \in G$, either $aH \cap bH = \emptyset$ or $aH = bH$.