

Review for Exam 2

Math 132, Spring 2009
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Disclaimer: This is not meant to be an exhaustive guide of everything you need to study for your midterm. It may not even be a complete list of topics from all of the sections it covers. This is simply a handout to aid your preparation and to remind you of topics you need to review.

This year, Exam 2 covers methods of integration, approximate integration, and improper integrals.

Remember to look back at the old exams. If nothing else, it's a good idea to practice looking at an integral and quickly making a choice of a good method to attempt first.

1 Symbolic Integration

The topic of symbolic integration covers our methods to approach a proper integral, from sections 5.5 and 7.1 – 7.5. You should read the strategies in section 7.5 for more ideas.

1.1 u -Substitution

The first topic we covered was u -substitution. When you see a familiar function with a complicated input, and you notice that the derivative of that complicated input is hanging around in the integrand somewhere, try substitution. You let u equal that complicated input. Then du will involve the dx term and the derivative of the complicated input. Be sure to replace any stray x 's with u 's!

If the integral is indefinite, then from here, you simply evaluate the new (easier!) integral, get an answer in terms of u , and then replace all u 's with the original substitution.

If the integral is definite, you have two options. You can change the limits of integration from values of x to values of u and then continue as usual. If you choose this method, in the end you simply plug in the values (you don't have to return to x !). Your other option is to evaluate the integral as though it was an indefinite integral, find the answer in terms of u , put it back in terms of x , and then put the answer back into the definite calculation.

Example 1.1 $\int_0^{\sqrt[3]{\pi}} x^2 \cos(x^3) dx$

We have the familiar function cosine with a complicated input, and the derivative of the complicated input is part of the integrand as well. We use substitution. Let $u = x^3$. Then $du = 3x^2 dx$ so $\frac{1}{3} du = x^2 dx$.

Method 1: When $x = 0$, we get $u = 0^3 = 0$, and when $x = \sqrt[3]{\pi}$, we get $u = (\sqrt[3]{\pi})^3 = \pi$. So we have

$$\int_0^{\sqrt[3]{\pi}} x^2 \cos(x^3) dx = \int_0^{\pi} \cos(u) du = \sin(u) \Big|_0^{\pi} = \sin(\pi) - \sin(0) = 0 - 0 = 0$$

Method 2: We treat this as an indefinite integral, until we get it back in terms of x . So we have:

$$\int x^2 \cos(x^3) dx = \int \cos(u) du = \sin(u) + C = \sin(x^3) + C$$

and so

$$\int_0^{\sqrt[3]{\pi}} x^2 \cos(x^3) dx = \sin(x^3) \Big|_0^{\sqrt[3]{\pi}} = \sin((\sqrt[3]{\pi})^3) - \sin(0^3) = \sin(\pi) - \sin(0) = 0.$$

1.2 Integration by Parts

The next method of integration that we learned was integration by parts. You want to use this method when the integrand is either a product of two functions that don't have a nice relationship (so u -substitution won't work),

or when the integrand is a single function for which you don't know (or don't remember!) a nice antiderivative (for example, $\ln(x)$).

You especially want to use this method when the integrand is a power of x or a polynomial multiplied by a trig or log function (a transcendental function).

For this method, we choose a piece of the integrand that has a nice derivative to be u and we take the rest of the integrand *including the dx* to be dv . We can then replace our original integral with the function $uv - \int v du$.

If you try integration by parts and you find that the integral you get is more complicated than the integral that you started with, you should go back and change your choice of parts.

We have a trick for the order in which to choose our u with this method. Choose u in order from the list:

- Inverse Trig Functions
- Logarithms
- Algebraic Expressions (polynomials, roots, etc.)
- Trig Functions
- Exponential Functions.

Example 1.2 $\int x^2 \arcsin(x) dx$

Here, we have a product of a nice function and an inverse trig function. We choose our parts using the list above. We have an inverse trig function, so that will be our u .

Let

$$\begin{aligned} u &= \arcsin(x) & dv &= x^2 dx \\ du &= \frac{1}{\sqrt{1-x^2}} dx & v &= \frac{x^3}{3} \end{aligned}$$

So

$$\int x^2 \arcsin(x) dx = \frac{x^3}{3} \arcsin(x) - \int \frac{x^3}{3} \frac{1}{\sqrt{1-x^2}} dx$$

The second integral is now easier than the first: we can use u -substitution to solve it.

Let $U = 1 - x^2$. Then $dU = -2x dx$ and $x^2 = 1 - U$, so

$$\begin{aligned}\int \frac{x^3}{3} \frac{1}{\sqrt{1-x^2}} dx &= \frac{1}{3} \int \frac{1-U}{\sqrt{U}} du = \frac{1}{3} \left(\int \frac{1}{\sqrt{U}} dU - \int \sqrt{U} dU \right) \\ &= \frac{1}{3} \left(\frac{\sqrt{U}}{\frac{1}{2}} - \frac{U^{\frac{3}{2}}}{\frac{3}{2}} \right) = \frac{2}{3} \sqrt{1-x^2} - \frac{2}{9} (1-x^2)^{\frac{3}{2}} + C\end{aligned}$$

And so:

$$\int x^2 \arcsin(x) dx = \frac{x^3}{3} \arcsin(x) - \left(\frac{2}{3} \sqrt{1-x^2} - \frac{2}{9} (1-x^2)^{\frac{3}{2}} \right) + C$$

1.3 Trig Integrals

The next topic we covered was strategies for integrating products of powers of trig functions. For this type of integral, we need to know the following trig identities.

1. $\sin^2(x) + \cos^2(x) = 1$
2. $\tan^2(x) + 1 = \sec^2(x)$
3. $\sin(2x) = 2 \sin(x) \cos(x)$
4. $\sin^2(x) = \frac{1}{2} (1 - \cos(2x))$
5. $\cos^2(x) = \frac{1}{2} (1 + \cos(2x))$

The strategies in this section depend on the powers you have in the integrand, but the idea behind each is the same. You look for something you can “save”. This will be your du . You replace all other occurrences of that function using one of the first two identities above, then you let u replace the other function that appears.

Example 1.3 $\int \sin^4(x) \cos^3(x) dx$

Here, we see that the power of \cos is odd. So we can save one of these to become our du . Then we need to use the first identity from the list to replace the rest of the cosines and let $u = \sin(x)$.

$$\int \sin^4(x) \cos^3(x) dx = \int \sin^4(x) \cos^2(x) \cos(x) dx = \int \sin^4(x)(1 - \sin^2(x)) \cos(x) dx$$

Now, let $u = \sin(x)$ so that $du = \cos(x) dx$. So

$$\begin{aligned} \int \sin^4(x)(1 - \sin^2(x)) \cos(x) dx &= \int u^4(1 - u^2) du = \int u^4 - u^6 du \\ &= \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C \end{aligned}$$

There are many more examples of this type of integral on the past exams and in the book and in your homework.

1.4 Trig Substitution

Trig substitution should, in some sense, be your last line of defense. Before you start using trig sub to solve an integral, be sure to check that you can't solve it using something like u -substitution or partial fractions (see the next subsection).

If you've checked these other methods and they don't work, it's time to decide which trig sub you need. This depends on the form of the term you're starting with. Remember, the term you're starting with (usually under a square root, though not always in this form), will look something like the trig identity you'll need.

If you have a term of the form $a^2 - x^2$ (where a is a number), then you want to replace x with $a \sin(\theta)$. You will end up using the identity $1 - \sin^2(\theta) = \cos^2(\theta)$ to solve this problem. Don't forget to replace dx with $a \cos(\theta) d\theta$.

If you have a term of the form $a^2 + x^2$, then you want to replace x with $a \tan(\theta)$ and you will use the identity $1 + \tan^2(\theta) = \sec^2(\theta)$. Don't forget to replace dx with $a \sec^2(\theta) d\theta$.

If you have a term of the form $x^2 - a^2$, then you want to replace x with $a \sec(\theta)$ and you will use the identity $\sec^2(\theta) - 1 = \tan^2(\theta)$. Don't forget to replace dx with $a \sec(\theta) \tan(\theta) d\theta$.

Example 1.4 $\int \frac{1}{\sqrt{36-x^2}} dx$

Here, we want to replace x with $6 \sin(\theta)$ and dx with $6 \cos(\theta) d\theta$. So we have

$$\begin{aligned} \int \frac{1}{\sqrt{36-x^2}} dx &= \int \frac{6 \cos(\theta)}{\sqrt{36 - (6 \sin(\theta))^2}} dx = \int \frac{6 \cos(\theta)}{\sqrt{36 - 36 \sin^2(\theta)}} d\theta = \int \frac{6 \cos(\theta)}{\sqrt{36(1 - \sin^2(\theta))}} d\theta \\ &= \int \frac{6 \cos(\theta)}{\sqrt{36 \cos^2(\theta)}} d\theta = \int \frac{6 \cos(\theta)}{6 \cos(\theta)} d\theta = \int 1 d\theta = \theta + C \end{aligned}$$

BE VERY CAREFUL HERE!! The answer we have now is θ , not x !

Now, we draw a right triangle and label one of the acute angles θ . Using our initial substitution, we know that $\sin(\theta) = \frac{x}{6}$, so the leg of the triangle opposite θ has length x and the hypotenuse has length 6. The third leg of the triangle has length $\sqrt{36 - x^2}$.

In this case, since our answer is simply θ , we won't need the triangle. If we'd gotten a trig function in our answer, then we would need the triangle to determine its value.

Here, since we have $\sin(\theta) = \frac{x}{6}$, we find that $\theta = \arcsin\left(\frac{x}{6}\right)$.

So

$$\int \frac{1}{\sqrt{36-x^2}} dx = \arcsin\left(\frac{x}{6}\right) + C.$$

1.5 Partial Fractions

When we are trying to integrate a rational function, and we see that the denominator is a quadratic polynomial that we can factor nicely, then we want to use partial fractions.

Example 1.5 $\int \frac{1}{x^2+4x-5} dx$

Here, we see that the denominator can be factored: $x^2+4x-5 = (x+5)(x-1)$.

So we have

$$\frac{1}{x^2 + 4x - 5} = \frac{1}{(x + 5)(x - 1)} = \frac{A}{x + 5} + \frac{B}{x - 1}$$

We multiply the last two terms by the entire denominator. This leaves us with

$$1 = A(x - 1) + B(x + 5).$$

To solve for A and B , we plug in the two roots of the polynomial (1 and -5), one at a time.

When we plug in 1, we get $1 = A(0) + B(6)$, so $1 = 6B$ and $B = \frac{1}{6}$. When we plug in -5 , we get $1 = A(-6) + B(0)$, so $1 = -6A$ and $A = -\frac{1}{6}$.

Then

$$\int \frac{1}{x^2 + 4x - 5} dx = \int \frac{-\frac{1}{6}}{x + 5} + \frac{\frac{1}{6}}{x - 1} dx = -\frac{1}{6} \ln(|x + 5|) + \frac{1}{6} \ln(|x - 1|) + C.$$

2 Approximate Integration

This material was covered in section 7.7. We have three methods for approximating a definite integral. In each case, we are finding a numerical approximation for the definite integral. We never actually evaluate the integral. Sometimes, we use these methods of approximation when the function we are interested in comes from a physical experiment, and so it doesn't have a nice expression—we only know its values at certain intervals.

In each example, we use n subintervals, and we let a and b denote the lower and upper limits of integration, respectively. Then $\Delta x = \frac{b-a}{n}$, and the endpoints of the subintervals are labelled x_0, x_1, \dots, x_n . In general $\bar{x}_i = \frac{x_{i-1} + x_i}{2}$ is the midpoint of the interval $[x_{i-1}, x_i]$.

2.1 The Midpoint Rule

$$\int_a^b f(x) dx \approx M_n = \Delta x (f(\bar{x}_1) + \dots + f(\bar{x}_n)).$$

2.2 The Trapezoidal Rule

$$\int_a^b f(x) dx \approx T_n = \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)).$$

2.3 Simpson's Rule

Remember, n has to be even if we want to use Simpson's Rule. (The 4's and 2's alternate all the way through.)

$$\int_a^b f(x) dx \approx S_n = \frac{\Delta x}{3}(f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n))$$

For examples using these rules, see your textbook, your class notes, your homework, and some of the past exams.

3 Improper Integrals

We have to ask whether an integral converges if either the integrand is not continuous on the interval of integration or the interval of integration is infinite (that is, one or both of the limits of integration is infinite). When we ask if an integral converges, we are asking whether or not it represents a finite number.

If the integrand is not continuous, we break up the integral at the point of discontinuity. We then evaluate each piece separately, taking limits as in the following examples. This is the only way to check whether an integral converges.

Example 3.1 Does the integral $\int_1^{10} \frac{1}{\sqrt[3]{x-9}} dx$ converge?

Notice that the integrand is not continuous at $x = 9$. Also,

$$\int \frac{1}{\sqrt[3]{x-9}} dx = \frac{3}{2} \sqrt[3]{(x-9)^2} + C.$$

Now,

$$\int_1^{10} \frac{1}{\sqrt[3]{x-9}} dx = \int_1^9 \frac{1}{\sqrt[3]{x-9}} dx + \int_9^{10} \frac{1}{\sqrt[3]{x-9}} dx.$$

Since

$$\int_1^9 \frac{1}{\sqrt[3]{x-9}} dx = \lim_{t \rightarrow 9^-} \int_1^t \frac{1}{\sqrt[3]{x-9}} dx = \lim_{t \rightarrow 9^-} \left(\frac{3}{2} \sqrt[3]{(x-9)^2} \right)_{x=1}^t = -\frac{3}{2} \sqrt[3]{(-8)^2} = -6$$

and

$$\int_9^{10} \frac{1}{\sqrt[3]{x-9}} dx = \lim_{b \rightarrow 9^+} \int_b^{10} \frac{1}{\sqrt[3]{x-9}} dx = \lim_{b \rightarrow 9^+} \left(\frac{3}{2} \sqrt[3]{(x-9)^2} \right)_{x=b}^{10} = \frac{3}{2} \sqrt[3]{(1)^2} = \frac{3}{2},$$

we see that $\int_1^{10} \frac{1}{\sqrt[3]{x-9}} dx$ converges, and is equal to $-6 + \frac{3}{2} = -4.5$.

Example 3.2 Does the integral $\int_0^\infty \frac{x}{(x^2+2)^2} dx$ converge?

First, $\int \frac{x}{(x^2+2)^2} dx = \frac{1}{2} \left(\frac{-1}{x^2+2} \right) + C$. So we have

$$\int_0^\infty \frac{x}{(x^2+2)^2} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{x}{(x^2+2)^2} dx = \lim_{t \rightarrow \infty} \left(\frac{1}{2} \left(\frac{-1}{x^2+2} \right) \right)_{x=0}^t = \frac{1}{4}.$$

Thus the integral converges.