A Few Words About Convergence

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Converge. The American Heritage Dictionary of the English Language, fourth edition, gives many definitions of the word *converge*. The first is "to tend toward or approach an intersecting point." The third is a mathematical definition: "to approach a limit."

Convergence. It also gives many definitions of the word *convergence*. The second is a mathematical definition: "the property or manner of approaching a limit, such as a point, line, function, or value."

When you are asked "Does (fill in the blank) converge or diverge?" the first thing you should ask is "What kind of creature is (blank)?"

In the last few weeks we've talked about convergence in three different settings. First, we talked about integrals converging. Then, we talked about sequences converging. Finally, we've been talking about convergence of series (infinite sums). In each situation, there are different ways to test for convergence.

Disclaimer. This is not meant to be a complete list of everything you need to study for your midterm. It may not even be a complete list of topics from the sections it covers. This is simply meant to be a self-contained study aid to help with understanding the concepts of convergence that we have been covering in class this spring.

1 Integrals

We have to ask whether an integral converges if either the integrand is not continuous on the interval of integration or the interval of integration is infinite (that is, one or both of the limits of integration is infinite). When we ask if an integral converges, we are asking whether or not it represents a finite number.

If the integrand is not continuous, we break up the integral at the point of discontinuity. We then evaluate each piece separately, taking limits as in the following examples. This is the only way to check whether an integral converges.

Example 1.1 *Does the integral* $\displaystyle\int_1^{10} \frac{1}{\sqrt[3]{x-9}}\,dx$ *converge?*

Notice that the integrand is not continuous at $x = 9$. *Also,*

$$\int \frac{1}{\sqrt[3]{x-9}}\,dx = \frac{3}{2}\sqrt[3]{(x-9)^2} + C.$$

Now,

$$\int_1^{10} \frac{1}{\sqrt[3]{x-9}}\,dx = \int_1^{9} \frac{1}{\sqrt[3]{x-9}}\,dx + \int_9^{10} \frac{1}{\sqrt[3]{x-9}}\,dx.$$

Since

$$\int_1^{9} \frac{1}{\sqrt[3]{x-9}}\,dx = \lim_{t\to 9^-}\int_1^{t} \frac{1}{\sqrt[3]{x-9}}\,dx = \lim_{t\to 9^-}\left(\frac{3}{2}\sqrt[3]{(x-9)^2}\right]_{x=1}^{t} = -\frac{3}{2}\sqrt[3]{(-8)^2} = -6$$

and

$$\int_9^{10} \frac{1}{\sqrt[3]{x-9}}\,dx = \lim_{b\to 9^+}\int_b^{10} \frac{1}{\sqrt[3]{x-9}}\,dx = \lim_{b\to 9^+}\left(\frac{3}{2}\sqrt[3]{(x-9)^2}\right]_{x=b}^{10} = \frac{3}{2}\sqrt[3]{(1)^2} = \frac{3}{2},$$

we see that $\displaystyle\int_1^{10} \frac{1}{\sqrt[3]{x-9}}\,dx$ *converges, and is equal to* $-6 + \frac{3}{2} = -4.5$.

Example 1.2 *Does the integral* $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$ *converge?*

First, $\int \frac{x}{(x^2+2)^2}\,dx = \frac{1}{2}(\frac{-1}{x^2+2}) + C$. *So we have*

$$\int_0^\infty \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\left(\frac{1}{2}\left(\frac{-1}{x^2+2}\right)\right]_{x=0}^{t} = \frac{1}{4}.$$

Thus the integral converges.

2 Sequences

Remember, a sequence is just a list of numbers in a specific order: "sequences have commas." When we ask if a sequence converges, we are asking whether the numbers in the list are getting infinitely close to a specific number.

Whenever possible, we try to find a pattern for the n^{th} term of the sequence, then take the limit of these terms as $n \to \infty$.

If this does not seem to work, we can use the squeeze theorem, look at the behavior of a function that defines the sequence and use L'Hospital's Rule, or take the limit of the absolute values of the terms of the sequence: if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Example 2.1 *To check convergence of the sequence* $\{ne^{-n}\}$*, we look at the behavior of the function* $f(x) = xe^{-x}$ *as* x *approaches infinity. We have*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0,$$

using L'Hospital's Rule. So $\lim\limits_{x\to\infty} f(x) = 0$*. Thus also* $\lim\limits_{n\to\infty} ne^{-n} = 0$ *and the sequence converges.*

Example 2.2 *We want to know whether the sequence* $\{\frac{\sin(n)}{n^2}\}$ *converges.*

When we see $\sin(n)$ *or* $\cos(n)$ *in the description of the terms of a sequence, we should first remember that for each* n, $-1 \le \sin(n) \le 1$ *and* $-1 \le \cos(n) \le 1$*. This should lead us to think about the Squeeze Theorem. Since* $-1 \le \sin(n) \le 1$*, also* $\frac{-1}{n^2} \le \frac{\sin(n)}{n^2} \le \frac{1}{n^2}$*. Now,* $\lim\limits_{n\to\infty} \dfrac{-1}{n^2} = 0 = \lim\limits_{n\to\infty} \dfrac{1}{n^2}$*.*

Thus, by the squeeze theorem, $\lim\limits_{n\to\infty} \dfrac{\sin(n)}{n^2} = 0$ *as well. So the sequence converges to* 0*.*

Example 2.3 *The sequence* $\{(-1)^n \frac{2n^2+3}{5n^3-3n+7}\}$ *converges to* 0*, since*

$$\begin{aligned}
\lim_{n\to\infty} \left| (-1)^n \frac{2n^2+3}{5n^3-3n+7} \right| &= \lim_{n\to\infty} \frac{2n^2+3}{5n^3-3n+7} \\
&= \lim_{n\to\infty} \frac{2n^2+3}{5n^3-3n+7} \frac{1/n^3}{1/n^3} \\
&= \lim_{n\to\infty} \frac{\frac{2}{n} + \frac{3}{n^3}}{5 - \frac{3}{n^2} + \frac{7}{n^3}} = 0.
\end{aligned}$$

Example 2.4 *The sequence* $\{\frac{3n^2+7n}{6-3n+n^2}\}$ *converges to* 3*, since*

$$\lim_{n\to\infty} \frac{3n^2+7n}{6-3n+n^2} = \lim_{n\to\infty} \frac{3n^2+7n}{6-3n+n^2}\frac{1/n^2}{1/n^2} = \lim_{n\to\infty} \frac{3+\frac{7}{n}}{\frac{6}{n^2}-\frac{3}{n}+1} = 3.$$

(Notice that n^2 *is the largest power of* n *that appears in the denominator.)*

3 Series

A *series* is an infinite sum. Remember: "series have plusses." Convergence of series is the most complicated type of convergence that we've studied so far.

There are many tests to determine whether a given series converges. Section 11.7 of the text outlines strategies for determining which test to use for a given series. Many of these tests involve checking limits of ratios of the terms or comparing the terms of the *sequence* whose terms are added to create the series. Some involve comparing the series to the integral of a related function.

When answering the question of whether a given series converges, be as specific as possible. Be sure to state which test you are using, and to show that any necessary conditions for use of that test are satisfied. Explain any notation that you create to answer the question.

In each of the following, we consider the series $\sum a_n$.

1. **Definition.** The n^{th} partial sum of the series is

$$s_n = \sum_{i=1}^{n} a_i.$$

   The series $\sum_{n=1}^{\infty} a_n$ is said to *converge* if $\lim_{n\to\infty} s_n$ exists. If $s = \lim_{n\to\infty} s_n$, then also $s = \sum_{n=1}^{\infty} a_n$.

2. **Test for Divergence.** If $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then the series $\sum a_n$ diverges. Remember: this limit does not tell us *anything* else! If $\lim a_n = 0$, we *must* use a different test.

3. **p-Series.** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

4. **Geometric Series.** The series $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$ converges if $|r| < 1$ and diverges if $|r| \geq 1$.

5. **Integral Test.** Suppose the function f is

   (a) continuous

   (b) positive, and

   (c) decreasing

   on the interval $[1, \infty)$, and $f(n) = a_n$ for each integer $n \geq 1$. The series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x)\, dx$ behave the same. That is:

   (a) If $\int_1^{\infty} f(x)\, dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.

   (b) If $\int_1^{\infty} f(x)\, dx$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

6. **Comparison Test.** You can use this test if $a_n > 0$ for each n and $b_n > 0$ for each n.

   Convergence: If $\sum b_n$ converges *and* $b_n \geq a_n$ for each n, then $\sum a_n$ converges.

   Divergence: If $\sum b_n$ diverges *and* $b_n \leq a_n$ for each n, then $\sum a_n$ diverges.

7. **Limit Comparison Test.** You can use this test if $a_n > 0$ for each n and $b_n > 0$ for each n. If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$, where c is a finite number and $c > 0$, then the series behave the same. That is, either both series $\sum a_n$ and $\sum b_n$ converge or both series diverge.

   (For both 6 and 7, usually $\sum a_n$ is the series we are asked to test for convergence or divergence, and we choose $\sum b_n$ to be a series we know about–most often a p-series or a geometric series.)

8. **Alternating Series Test.** An alternating series is one of the form $\sum(-1)^n b_n$ or of the form $\sum(-1)^{n-1} b_n$, where $b_n > 0$ for each n. If

   (a) $b_n \geq b_{n+1}$ for each n, and

   (b) $\lim_{n\to\infty} b_n = 0$,

   then the series converges.

   If one of these conditions is not satisfied, then you must use a different test.

9. **Absolute Convergence.** If $\sum |a_n|$ converges, then also $\sum a_n$ converges.

10. **Ratio Test.**

    (a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and therefore convergent.

    (b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

    (c) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then this test is inconclusive. You must try another test.

11. **Root test.**

    (a) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and therefore convergent.

    (b) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

    (c) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then this test is inconclusive. You must try a different test.

Also, don't forget that many of these tests come with methods to estimate the error involved in using the test to approximate a sum.

1. **Remainder Estimate for the Integral Test.** Suppose $f(n) = a_n$, where f is a continuous, positive, decreasing function for $x \geq k$ (for

some k), and $\sum a_n$ is convergent. If $R_k = s - s_k$, where s_k is the k^{th} partial sum, then

$$\int_{k+1}^{\infty} f(x)\,dx \leq R_k \leq \int_{k}^{\infty} f(x)\,dx.$$

We also have the inequality

$$s_k + \int_{k+1}^{\infty} f(x)\,dx \leq s \leq s_k + \int_{k}^{\infty} f(x)\,dx.$$

2. If we use the comparison test with the series $\sum b_n$ to show that the series $\sum a_n$ converges, and if $\sum b_n = t$, with partial sums $t_k = \sum_{i=1}^{k} b_i$, and $\sum a_n = s$, with partial sums $s_k = \sum_{i=1}^{k} a_i$, then we have the following inequality of remainders. Since $a_n \leq b_n$ for each n, if we let $R_k = s - s_k$ and $T_k = t - t_k$, then $R_k \leq T_k$.

3. **Alternating Series Estimation Theorem.** If $s = \sum(-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

   (a) $0 \leq b_{n+1} \leq b_n$, and

   (b) $\lim_{n\to\infty} b_n = 0$,

   then $|R_n| = |s - s_n| \leq b_{n+1}$.

3.1 How do you know which test to choose?

1. Check if the series is a p-series or a geometric series, since we know easily how to check whether or not these types of series converge. If not, go on to the next step.

2. If the series looks a little bit like a p-series or a geometric series, then use the Comparison Test or the Limit Comparison Test. Remember, the series $\sum a_n$ is the series you are given. You choose the series $\sum b_n$ by taking the dominant terms from the numerator and the denominator. If not, go on to the next step.

3. If it looks like $\lim_{n\to\infty} a_n \neq 0$, then use the Test for Divergence.

4. If the series has the form $\sum(-1)^n b_n$ or $\sum(-1)^{n-1} b_n$, then try the Alternating Series Test.

5. If your series involves strange products or factorials, try the Ratio Test. (But don't try this test for p-series or rational functions– it will be inconclusive in this case.) If the terms are raised to the n^{th} power, try the Root Test.

6. Finally, if the terms of your series look like a function, try the Integral Test.

If you get through the list, and you don't have an answer yet, go back and try again.