

## MATH 621 COMPLEX ANALYSIS, SAMPLE FINAL QUESTIONS

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You may use any theorem we proved in class, or from the book, that you can state precisely.

0. Review all questions from the sample midterm and midterm.

1. Evaluate  $\int_{\gamma} z \sin z^2 dz$ , where  $\gamma$  is the unit circle.

2. Prove the fundamental theorem of algebra using complex analysis.

3. Classify the singularities of the following functions, and, where appropriate, evaluate their non-zero residues:

$$a) f(z) = 1/(z^3 - 2) \quad b) g(z) = (\cos(z) - 1)/z^3, \quad c) h(z) = (z^2 - 9)^{1/2}.$$

4. Prove that for any  $a > 1$ , the equation  $z = e^{z-a}$  has exactly one solution within the unit circle. [hint: think of the roots of  $h(z) = z - e^{z-a}$ ].

5. For real  $a$ , evaluate

$$I = \int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2} dx.$$

6. Prove the Phragmen-Lindelöf Theorem: If  $f$  is holomorphic on an open set  $\Omega$  containing the strip

$$S = \{z \in \mathbb{C} : -1 \leq \Re(z) \leq 1\},$$

and  $|f| \leq K$  for some positive constant  $K$ , as well as  $|f| \leq 1$  on the boundary of  $S$ , then  $|f| \leq 1$  on all of  $S$ .

7. Find a conformal map from the region  $\{z \in \mathbb{C} : \Re(z) > 0, \Im(z) > 0, |z| < 1\}$  to itself which cyclically permutes the points  $0, 1, i$  (after extending the map to the closure by continuity).

8. Suppose  $(a_n)$  is a sequence of complex number of polynomial growth, i.e.  $|a_n| \leq n^t$  for some real number  $t \geq 0$ . Show that the Dirichlet series  $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$  converges absolutely for  $\Re(s) > t + 1$ .

9. Find all entire functions which satisfy the following conditions:

(i)  $|f''(z)| \leq 10|z|^2$  for all  $z$

(ii)  $f$  has a zero of order 3 at the origin

(iii)  $f(1) = 1$ .

10. Evaluate

$$\frac{1}{2\pi i} \int_{|w|=4} w^5 \frac{\cos w}{\sin w} dw.$$

11. Suppose  $f$  is holomorphic in a region containing the closed unit disc and that  $|f(z)| > 2$  if  $|z| = 1$ . Suppose further that  $f(0) = 1$ . Show that  $f$  has a zero inside the unit disc.

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There will probably be about  $n$  questions, where  $n \leq 10$  and you will be asked to answer  $n - 3$  of them.

12. Let  $f$  be holomorphic in a region containing the closed unit disc. Suppose  $0 < |f(z)| < 1$  if  $|z| = 1$ . Show that  $f$  has exactly one fixed point inside the unit disc. A point  $z_0$  is fixed by  $f$  if  $f(z_0) = z_0$ .
13. State and prove the Schwarz Lemma.