MATH 621 COMPLEX ANALYSIS, HOMEWORKS 6 AND 7

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Here are some problems I would like you to solve in the second half of the term. You now have all the tools needed to solve them, but they will not all be due at once. I will announce in class on Monday March 26 after Spring Break which ones will be due for Friday March 30, and which for the following Friday April 6.

I. Stein-Shakarchi Chapter 2: 9, 14, 15, and Chapter 3: 2,3,13,14,15,19.

II.

1. Suppose C_1, C_2 are simple closed curves in the complex plane. Let Ω_1 be the region **inside** C_1 and let Ω_2 be the region **outside** C_2. Assume that C_2 lies entirely in Ω_1, and let $\Omega = \Omega_1 \cap \Omega_2$ be the "quasi-annular" region between the two curves. Show that any holomorphic function f on Ω can be decomposed as a sum $f = f_1 + f_2$ where f_i is holomorphic on Ω_i for $i = 1, 2$. Moreover show that this decomposition is unique up to an additive constant, i.e. if g_i are holomorphic on Ω_i and add up to f, then there is a constant c such that $f_1 = g_1 + c$ and $f_2 = g_2 - c$.

2. Give the Laurent series expansions for the regions indicated:

a)

$$e^{1/(z-1)}, |z| > 1$$

b)

$$\frac{1}{(z-a)(z-b)}, 0 < |a| < |z| < |b|$$

c) same function as b) but for the region $|z| > b$.

Some Definitions about Behavior of functions at ∞.

How can we define a "neighborhood" of infinity? We use the intuitive idea that for z to be close to infinity, $1/z$ has to be close to 0. Thus, for a real number $R > 0$, we define a disc of radius R about ∞ to be $D_R(\infty) = \{z \in \mathbb{C} : |z| > R\}$. To be "close to ∞," therefore, means that you are outside the circle of radius R centered at 0 for a large R. Equivalently, for $z \in \mathbb{C}$, $z \in \mathcal{D}_R(\infty)$ if and only if $1/z \in D_r(0)$ where $r = 1/R$.

Here is a rough definition, to be made precise below. We say that a function f behaves a certain way at ∞ if the function $g(z) = f(1/z)$ behaves that way at $z = 0$.

So, let us suppose that f is defined on $D_R(\infty)$ for some $R > 0$. We say that f is holomorphic at ∞ if there exists a real number $r > 0$ such that $g(z) = f(1/z)$ extends to a holomorphic function on $D_r(0)$. In that case, we write $f(\infty) := g(0)$ which is of course $\lim_{z\to 0} g(z)$. We say f has a pole of order n at ∞ if $g(z) = f(1/z)$ extends to a holomorphic function on $D_r^0(0)$ with a pole of order n at 0. More generally, if $g(z) = f(1/z)$ is holomorphic on the punctured neigborhood $D_r^0(0)$, then the type of the singularity of f at ∞ is determined by the Laurent series expansion of $g(z)$ at $z = 0$.

For example, since $e^{1/z}$ has an essential singularity at $z = 0$, we say that function e^z has an essential signularity at $z = \infty$.

3. Determine the type of singularity each function have at the indicated point; show your work.

a)
$$\sin\left(\frac{1}{z-1}\right), z = 1$$

b)
$$\frac{1}{1-e^z}, z = 2\pi i$$

c)
$$\frac{1}{\sin(z) - \cos(z)}, z = \pi/4$$

The rest are at $z = \infty$

d)
$$\frac{z^2+4}{e^z}$$

e)
$$\cos(z) - \sin(z)$$

f)
$$\cot(z)$$

g)
$$e^{-1/z^2}$$

4. Here is a fact I am not asking you to prove, but you may do so for extra credit (it's not particularly hard):

(*) Suppose $z_1, \ldots, z_k \in \mathbb{C}$ and $\alpha_1, \ldots, \alpha_k$ is a "partition of unity" i.e. they are non-negative reals adding to 1, $\alpha_1 + \cdots + \alpha_n = 1$, then the number $\zeta = \sum_{\nu=1}^{k} \alpha_\nu z_\nu$ lies in the "convex hull" of the z_ν's (the smallest closed convex polygon containing these points). Moreover, if z is any complex number satisfying
$$\sum_{\nu=1}^{k} \frac{\alpha_\nu}{z - z_\nu} = 0,$$
the z lies in the convex hull of $z_1, \ldots, z_k$.

(a) What I am asking you to prove using (*) is:

If $P(z)$ is a non-constant complex polynomial with roots $z_1, \ldots, z_k$, then the roots of $P'(z)$ are contained in the convex hull of $z_1, \ldots, z_k$. This is sometimes called the Gauss-Lucas Theorem.

(b) Show that if the roots of a complex polynomial $P(z)$ are real, then so are the roots of all of its derivatives.

5. (a) Suppose f has a zero of order n at z_0. What is the residue of
$$z\frac{f'(z)}{f(z)} \text{ or more generally } \varphi(z)\frac{f'(z)}{f(z)}$$
at $z = z_0$. Here $\varphi(z)$ is any function which is holomorphic at z_0.

(b) State (with the requisite assumptions on f, C and φ) and prove a generalization of the Argument Principle for the integrals

$$\frac{1}{2\pi i}\int_C z\frac{f'(z)}{f(z)}dz, \qquad \frac{1}{2\pi i}\int_C \varphi(z)\frac{f'(z)}{f(z)}.$$

6. Suppose that f is holomorphic on an open set Ω containing the closed unit disc such that $|f(z)| < 1$ whenever $|z| = 1$. Show that the equation $f(z) = z^3$ has exactly three solutions (counting multiplicities) inside the unit disc.

7. How many zeros does the function $f(z) = 3z^{621} - e^z$ have inside the unit disc (counting multiplicities)? Do they all have multiplicity 1?

8. (a) Suppose f is an entire function such that $|f(z)| \leq 621|\cos(z)|$ for all $z \in \mathbb{C}$. Show that $f(z) = k\cos(z)$ for some constant k of absolute value at most 621.

Hint: Riemann to the rescue!

(b) Generalize (a).

(c) Explain how you could try to nail down the exact location of a simple zero of a holomorphic function using integration if you had a good estimate on where the zero is. Do you think this would be as practical as say Newton's method for finding real roots of real polynomials?