MATH 621 COMPLEX ANALYSIS
SAMPLE MIDTERM EXAM

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Short answer:

(1) True or False: If F is any field, then $F[x_1, \ldots, x_{2007}]$ is a UFD.
(2) True or False: $\mathbb{Z}[x]$ is a PID.
(3) True or False: $\mathbb{Z}[x]$ is a UFD.
(4) Let $a(x) = x^7 - 1$ and $b(x) = x^3 - 1 \in F[x]$ where $F = \mathbb{Z}/5\mathbb{Z}$. Compute $q(x)$ and $r(x)$ in $F[x]$ such that $a(x) = q(x)b(x) + r(x)$ where $\deg(r) < \deg(b)$.

Not-necessarily-short answer:

(1) Show that a finite integral domain is a field.
(2) Suppose R, R' are commutative rings, I is an ideal of R and $\varphi : R \to R'$ is a ring homomorphism. Let $I' = \varphi(I)R'$ be the set of R'-multiples of $\varphi(I)$. Prove or Disprove: I' is an ideal of R'.
(3) Let R be a commutative ring, and suppose a is a nilpotent element of R. Show that $1 - a$ is a unit of R.
(4) Let $\varphi : R \to R'$ be a surjective homomorphism of commutative rings, and let M be a maximal ideal of R. Show that $\varphi^{-1}(M) := \{r \in R \mid \varphi(r) \in M\}$ is a maximal ideal of R. (Don't forget to show that $\varphi^{-1}(M)$ is an ideal of R.)
(5) Give a complete non-redundant list of ideals of the ring $\mathbb{Z}/24\mathbb{Z}$.
(6) Let $\varphi : R \to R'$ be a surjective ring homomorphism.
   (a) Prove or disprove: If R is an integral domain, then R' is an integral domain also.
   (b) Prove or disprove: If R' is an integral domain, then R is also an integral domain.
(7) Prove or disprove: there exists a field of characteristic 2007.
(8) Show that if F is a field, and R is a ring, then every non-zero homomorphism $F \to R$ is injective.
(9) A commutative ring R is called a *local ring* if it has exactly one maximal ideal. Prove that if R is a local ring with maximal ideal M, then every element of of $R - M$ is a unit of R.
(10) Prove that the ring of $R = \mathbb{Z}_{(2)}$ all rational numbers with odd denominator is a local ring with maximal ideal $(2) = 2R$. We call this ring "$\mathbb{Z}$ localized at 2."
(11) State and prove Gauss' Lemma.
(12) Show that an ascending chain of ideals in a PID is eventually constant.