UMASS AMHERST MATH 471 FALL 2006, F. HAJIR

HOMEWORK 9: ARITHMETIC FUNCTIONS

Let us make a couple of definitions first. An *arithmetic function* is a function $f : \mathbb{Z}^+ \to \mathbb{C}$ where $\mathbb{Z}^+$ is the set of positive integers. For example, the Euler phi function $\varphi : \mathbb{Z}^+ \to \mathbb{Z}^+$ sending m to $\varphi(m) = |(\mathbb{Z}/m\mathbb{Z})^\times|$ is an arithmetic function. We say that an arithmetic function f is *multiplicative* if

$$f(mn) = f(m)f(n) \text{ for all coprime integers } m, n \in \mathbb{Z}^+.$$

Please note the condition that the multiplicativity condition $f(mn) = f(m)f(n)$ does not need to hold for all pairs m, n, only for those that satisfy $\gcd(m, n) = 1$. One way to think of this is that a multiplicative function f is "free" to hold whatever values it wants on the prime powers $m = p^t$, but once those are chosen, then all the remaining values are determined by the fact that each integer is (uniquely) a product of primes.

Given an arithmetic function $f : \mathbb{Z}^+ \to \mathbb{C}$, we construct the *divisor-sum function of f*, called F, by

$$F(m) = \sum_{d|m} f(d).$$

Here, by convention, the notation "$d|m$" indicates that the sum is over the set of *positive* divisors d of m.

Some other standard arithmetic functions are the nu-function $\nu(m) = |\mathrm{Div}^+(m)|$ which counts the number of divisors of m and the sigma-function $\sigma(m) = \sum_{d|m} d$ which gives the *sum* of the divisors of m.

A very useful (but at first sight really weird) function is the Möbius mu (pronounced "mew") function μ. It can be defined as follows. We have $\mu(1) = 1$ (see problem 1 below for why this is a popular choice) and put $\mu(2) = \mu(3) = \mu(5) = \mu(7) = \mu(11) = \cdots = -1$, in fact let's put $\mu(p) = -1$ for every prime p. What about prime powers? Here you might be surprised by the choice I am about to make: If $t > 1$, and p is a prime, we define $\mu(p^t) = 0$. So, μ "kills" higher prime powers. Finally, we define μ on the rest of the integers by multiplicativity. To put this all together, the definition of μ is as follows. First, $\mu(m) = 1$ if $m = 1$, and $\mu(m) = (-1)^r$ if $m = p_1 \cdots p_r$ is the product of r *distinct* primes $p_1, \ldots, p_r$. But if m is divisible by the square of any prime, then $\mu(m) = 0$. You'll see below why this is a useful function.

1. Show that if f is a multiplicative arithmetic function, then $f(1) = 1$.

2. Suppose $\mathrm{Div}^+(m) = \{1 \leq d \leq m \mid d \text{ is a divisor of } m\}$ is the set of positive divisors of an integer m. Show that if m, n are coprime, then

$$\mathrm{Div}^+(mn) = \{dd' \mid d \in \mathrm{Div}^+(m) \text{ and } d' \in \mathrm{Div}^+(n)\}.$$

Hint: Show that the LHS is contained in the RHS and versa vice.

3. Show that if f is a multiplicative arithmetic function, with divisor-sum function F, then F is also a multiplicative arithmetic function.

Hint: It boils down to justifying

$$\left(\sum_{d|m} f(d)\right)\left(\sum_{d'|n} f(d')\right) = \sum_{d|m}\sum_{d'|n} f(d)f(d') = \sum_{e|mn} f(e),$$

for which verification you use 2.

4. In this problem, we will prove that for an integer $m \geq 1$,

$$\sum_{d|m} \varphi(d) = m, \tag{1}$$

where, as before, the notation "$d|m$" indicates that the sum is over the set of *positive* divisors d of m. Let $\Phi(m) = \sum_{d|m} \varphi(d)$ be the divisor-sum function of φ. We want to prove that $\Phi(m) = m$.

(i) Verify the formula (1) for $1 \leq m \leq 10$, recall that $\varphi(1) = 1$ by definition.

(ii) Verify that $\Phi(m) = m$ when $m = p^t$ is a power of a prime p. (Don't forget the all-purpose identity $(x-1)(1+x+x^2+\cdots+x^t) = x^{t+1}-1$).

(iii) Prove that $\Phi(m) = m$ holds for all m by using (ii) and Problem 3.

5. (**The Möbius Inversion Formula**) Suppose f is a multiplicative arithmetic function, with divisor-sum function F, so that $F(m) = \sum_{d|m} f(d)$. Prove that

$$f(m) = \sum_{d|m} \mu(m/d)F(d)$$

by following the steps below. Let $g(m) = \sum_{d|m} \mu(m/d)F(d)$.

(i) Check that g is a multiplicative function.

(ii) Check that $g(p^t) = f(p^t)$ for a prime p and an integer $t \geq 0$.

(iii) Use (i) and (ii) to conclude that $g(m) = f(m)$ for all m.

6. Show that for $m \geq 1$, $\varphi(m) = m\sum_{d|m} \mu(d)/d$.

7. (i) Show that if $m = p_1^{t_1}\cdots p_r^{t_r}$ is the prime factorization of m, then $\nu(m) = (t_1+1)\cdots(t_r+1)$. Hint: you don't need any fancy machinery for this, just consider the prime factorization of the possible divisors of m and use the "fundamental counting principle").

(ii) Show that $\nu(m)$ is odd if and only if m is a perfect square. (Again, you might want to think about this in elementary terms).

8. Show that for $m \geq 1$,

(i) $\sum_{d|m} \mu(n/d)\nu(d) = 1$,

(ii) $\sum_{d|m} \mu(m/d)\sigma(d) = m$.