UMASS AMHERST MATH 471 FALL 2006, F. HAJIR

SOLUTIONS FOR HOMEWORK 4: PRIME DISTRIBUTION I

1. Show that in the sequence of primes, there are arbitrarily large gaps. To be more precise, let N be any positive integer. Show that there exists an integer a such that

$$a+1, a+2, a+3, \ldots, a+N$$

are all composite. (An integer is composite if it is greater than 1 and not prime). [1]

Take $b = (N+1)! > 1$. Then 2 divides b so 2 divides $b+2$. Since $b > 1$, $b+2$ is composite. Also 3 divides b so 3 divides $b+3$. Since $b > 1$, $b+3$ is composite. Similarly, for $2 \leq j \leq N+1$, $j|b$ hence $j|(b+j)$ and $b+j > j$ so $b+j$ is composite. Thus, $b+2, b+3, \ldots, b+(N+1)$ are N consecutive composites.

2. Let $\pi(x) = |\{2 \leq p \leq x \mid p \text{ is prime }\}|$ be the "prime-counting" function. Its value at a real number x is the number of primes not less than x. Show that $\pi(n) \leq n/2$ for all integers $n \geq 8$.

We do this by complete induction. We let $S(n)$ be the statement $\pi(n) \leq n/2$. First of all, $\pi(8) = 4 \leq 8/2$ and $\pi(9) = 8 \leq 9/2$ so the statement $S(n)$ in the problem is true for $n = 8$ and $n = 9$. Suppose $S(i)$ is true for $8 \leq i \leq k$ where k is some integer satisfying $k \geq 9$. In particular, we suppose that $S(k)$ and $S(k-1)$ are true. If $k+1$ is composite, then $\pi(k+1) = \pi(k) \leq k/2 < (k+1)/2$ so $S(k+1)$ holds. If $k+1$ is prime, then k must be even, so k cannot be a prime since $k > 2$. Thus, $\pi(k) = \pi(k-1)$. But $\pi(k-1) < (k-1)/2$ because by assumption $S(k-1)$ holds. Thus, $\pi(k+1) = \pi(k-1) + 1 < (k-1)/2 + 1$, i.e. $\pi(k+1) < (k+1)/2$ which is just $S(k+1)$. Thus, in all cases we have proved that $S(k-1)$ and $S(k)$ together imply $S(k+1)$. By the principle of mathematical induction, $S(n)$ is true for all $n \geq 8$.

3. Show that $\pi(n) \leq n/3$ for all integers $n \geq 33$.

This is similar to the above. We let $S(n)$ be the statement that $\pi(n) \leq n/3$.
First we check that it's true for $n = 33$ and $n = 34, 35, 36$, I'll leave that to you. Then we want to show that for $k \geq 34$, if $S(i)$ holds for all $33 \leq i \leq k$, then $S(k+1)$ holds.
Again, there are various cases. First, if $k+1$ is composite, then $\pi(k+1) = \pi(k) \leq k/3 < (k+1)/3$ so $S(k+1)$ holds. Next, we will assume that $k+1$ is prime. Then $k+1$ is odd so k is even and cannot be prime. We have then $\pi(k+1) = 1 + \pi(k-1) \leq 1 + (k-1)/3 \leq$

[1]Hint: one way to do this is to write $a = b+1$, so the sequence we want is $b+2, b+3, \ldots, b+(N+1)$. Now let's try to arrange so that $b+2$ is divisible by 2, $b+3$ is divisible by $3, \ldots$, $b+(N+1)$ is divisible by $N+1$. What does this mean about b? If this drives you crazy and you want a further hint, send an e-mail to Farshid.

$(k+2)/3$. Now we can write $k = 3j + r$ with $r = 0, 1$ or 2. We eliminate $k = 3j + 2$ immediately because $k+1$ is not a multiple of 3. If $k = 3j$, then $(k+2)/3 = j + 2/3$ and $\pi(k+1) \leq j + 2/3$ implies that $\pi(k+1) \leq j$ because $\pi(k+1)$ is an integer. This gives $\pi(k+1) \leq k/3 < (k+1)/3$ which is what we wanted. There remains the case where $k = 3j+1$. In this case, $k-1 = 3j$ cannot be a prime, so $\pi(k-1) = \pi(k-3) \leq (k-3)/3$. We then get $\pi(k+1) = 1 + \pi(k-3) \leq 1 + (k-3)/3 < (k+1)/3$. This covers all the cases so we are done.

4. For an integer $N \geq 2$, define

$$\zeta_N(x) = \prod_{p \leq N} \frac{1}{1 - \frac{1}{p^x}}.$$

Recall the convention that $\prod_p$ always refers to a product over the prime numbers, thus $\prod_{p \leq N}$ means the product over the primes not exceeding N. Also, let $f(N) = \zeta_N(1)$.

(a) Fix an integer N and let S be the set of positive integers that are expressible as products of primes not exceeding N. Thus, if the primes not exceeding are $p_1, \ldots, p_s$, then

$$S = \{p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} \mid a_i \geq 0, i = 1, \ldots, s\}.$$

Note that $1 \in S$ because $1 = \prod_{p \leq N} p^0$. Prove that

$$\zeta_N(x) = \sum_{n \in S} \frac{1}{n^x}.$$

Plug in $x = 1$ to get $f(N) = \sum_{n \in S} 1/n$.

Hint: use the same proof that we used in class for $\zeta(x)$ by expanding $1/(1 - 1/p^x)$ in a geometric power series.

We expand each $1/(1 - 1/p_i^x) = 1 + p_i^{-x} + p_i^{-2x} + p_i^{-3x} + \cdots$. When we need to, we'll think of p_i^{-mx} as $(p_i^m)^{-x}$. When we multiply these s infinite series together, an arbitrary term in the sum is

$$(p_1^{a_1})^{-x}(p_2^{a_2})^{-x} \cdots (p_s^{a_s})^{-x} = (p_1^{a_1} \cdots p_s^{a_s})^{-x},$$

where $a_1, a_2, \ldots, a_s$ is allowed to be any sequence of s non-negative integers. By the uniqueness of factorization and by the definition of the set S, each number in S can be expressed as such a product of primes in exactly one way, thus

$$\zeta_N(x) = \sum_{n \in S} n^{-x}.$$

We plug in $x = 1$ into this to get $f(N) = \sum_{n \in S} n^{-1}$.

(b) Prove that

$$\lim_{N \to \infty} f(N) = \infty.$$

You may assume the fact we proved in class that the harmonic series diverges.

As N goes to infinity, the set S converges to the set of all positive integers and so $\lim_{N \to \infty} f(N) = 1 + 1/2 + 1/3 + \cdots = \infty$ because the harmonic series diverges. But the definition of $\zeta_N(x)$ would say that for all N

(c) Show how (b) and the fact that the harmonic series diverges proves that there are infinitely many primes.

If there were finitely many primes, say p_s was the largest prime, then once we choose $N = p_s$, the set S would be all the positive integers, so $f(N) = \zeta_N(1) = 1 + 1/2 + 1/3 + \cdots = \infty$. But from the definition of $\zeta_N(x)$, $f(N) = \zeta_N(1) = \prod_{i=1}^{s} 1/(1 - 1/p_i^x)$ is just a bounded real number. This is a contradiction.

5. Suppose k, r are integers satisfying $k \geq 0$ and $0 \leq r \leq 5$.
(a) Show that if $p = 6k + r > 3$ is a prime, then $r = 1$ or $r = 5$.

Recalling that $p > 3$, we see that if $r = 0, 2$ or 4, then $p = 6k + r$ is even, so cannot be a prime. And if $r = 3$, then $p = 6k + 3$ is a multiple of 3 hence cannot be a prime. Thus, we must have $r = 1$ or 5.

(b) Show that the set $\{6k + 1 \mid k \geq 0\}$ is closed under multiplication.

Easy: If $k, l \geq 0$, $(6k+1)(6l+1) = 36kl + 6(k+l) + 1 = 6m + 1$ where $m = 6kl + k + l \geq 0$.

(c) Show that the set $\{6k + 5 \mid k \geq 0\}$ contains at least one prime. Then show that it contains infinitely many primes.

Well, the prime number 5 is in there. Suppose this set had only finitely many primes, call them $p_1, \ldots, p_s$. Let $N = 6p_1 \cdots p_s - 1 > 29$. Since $N > 1$, we know that it has at least one prime divisor. We know that 2 and 3 do not divide N. We also know that its prime divisors cannot *all* be of the form $6k + 1$ because by part (b) it would then follow that $N = 6k + 1$ for some integer k, but then $N = 6k + 1 = 6m - 1$ with $m = p_1 \cdots p_s$ implies that $6|2$ which is absurd. So, we have proved that N has at least one prime divisor p of the form $p = 6j + 5$. But we listed all the primes of this form, $p = p_i$ for some i, $1 \leq i \leq s$. This is absurd, however, because N/p_i is not an integer. This absurdiy completes our proof.

(d) Find integers b, r with $b \neq 2, 3, 6$ such that you can show the arithmetic progression $b + r, 2b + r, 3b + r, \ldots$ contains infinitely many primes in the same way.

We can do this with $b = 4$ and $r = 3$, as we did in class.