UMASS AMHERST MATH 471 FALL 2006, F. HAJIR

HOMEWORK 3: GAUSSIAN INTEGERS III

This homework is due at the start of class on Monday Oct 3. There will be no extension for this homework.

Reminder: Office hours are Monday and Wednesday at 11 in 1118 LGRT.

NOTE: THIS ASSIGNMENT IS NOW COMPLETE; THERE WON'T BE ANY PROBLEMS ADDED TO THIS ASSIGNMENT.

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1. Determine a generator for the ideal $(\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_1 = -1 + 3i$, $\alpha_2 = -4 + 7i$, and $\alpha_3 = -3 + 4i$.

2. (a) Using the Euclidean algorithm, compute a generator γ for the ideal $(-4+7i, -3+4i)$ and find $\lambda, \mu \in \mathbb{G}$ such that $\gamma = \lambda(-4 + 7i) + \mu(-3 + 4i)$.
(b) Using the Euclidean algorithm, compute a generator γ for the ideal $(-11 + 49i, -33 + 56i)$ and find $\lambda, \mu \in \mathbb{G}$ such that $\gamma = \lambda(-11 + 49i) + \mu(-33 + 56i)$.

3. Suppose $\alpha, \beta \in \mathbb{G}$ and $\gcd(|\alpha|^2, |\beta|^2) = 1$. Prove that $(\alpha, \beta) = \mathbb{G}$.

4. Prove or disprove: Whenever $\alpha, \beta, \gamma \in \mathbb{G}$ where $(\gamma) = (\alpha, \beta)$, then $|\gamma|^2$ divides $\gcd(|\alpha|^2, |\beta|^2)$.

5. Suppose $\alpha, \beta, \pi \in \mathbb{G}$ and π is a Gaussian prime. Show that if $(\alpha, \beta) = \mathbb{G}$ and $\pi|\alpha\beta$ (i.e. there exists $\delta \in \mathbb{G}$ such that $\pi\delta = \alpha\beta$), then either $\pi|\alpha$ or $\pi|\beta$.

6. For $\alpha, \beta, \gamma \in \mathbb{G}$ show that if $(\alpha, \beta) = \mathbb{G}$, then $(\gamma\alpha, \gamma\beta) = (\gamma)$.

7. (a) Show that for $a, b \in \mathbb{Z}$, if $\gcd(a, b) = 1$ then $\gcd(a + b, a - b)$ is either 1 or 2.
(b) For $\alpha, \beta \in G$ such that $(\alpha, \beta) = \mathbb{G}$, what values can $\gcd(\alpha + \beta, \alpha - \beta)$ take? Prove that your answer is correct.

8. Suppose $(\alpha, \beta) = (\gamma)$. Thus, γ divides α and β, so we can write $\gamma\alpha' = \alpha$ and $\gamma\beta' = \beta$ for some $\alpha', \beta' \in \mathbb{G}$. Prove that $(\alpha', \beta') = \mathbb{G}$.
Hint: One option is to use proof by contradiction.

9. Suppose $\alpha, \beta, \gamma \in \mathbb{G}$, $(\alpha, \beta) = \mathbb{G}$ and $\alpha\beta = \gamma^2$. Show that there exist $\lambda, \mu \in \mathbb{G}$ such that $\alpha = \lambda^2, \beta = \mu^2$.
Hint: Use the fact that $\mathbb{G}$ is a unique factorization domain.

Extra Credit

A. Prove that given $\alpha, \beta \in \mathbb{G}$, there exist $\lambda \in \mathbb{G}$ (unique up to multiplication by a unit) such that $(\alpha) \cap (\beta) = (\gamma)$. Explain why γ should be called a *lowest common multiple* for α and β.

B. With α, β, λ as in A, and $(\alpha, \beta) = (\gamma)$ prove that $\alpha\beta = \gamma\lambda\varepsilon$ for some unit $\varepsilon \in \mathbb{G}^{\times}$.