

UMASS AMHERST MATH 471 FALL 2006, F. HAJIR

EXAM 1 SAMPLE

Exam 1 will take place Monday Oct 16 in class. Don't be late. The format of the exam will be pretty close to what you get here (approximately the same number of questions, etc.)

For the true/false and short answer sections, you do not need to justify your answer, just GET the answer. For the "proofs" part, you do need to justify your steps.

Don't spend too much time on the short answer/definitions/TF. You want to leave enough time to think about the proofs which carry half the points roughly.

1. Define the following terms. Unless otherwise noted, all Greek letters denote elements of the set of Gaussian integers.

- a. The set of Gaussian integers is
- b. A Gaussian integer α is called a *unit* if
- c. To say that $(\gamma) = \gcd(\alpha, \beta)$ means
- d. $\alpha|\beta$ means
- e. The norm $N(\alpha)$ of a Gaussian integer α is defined to be
- f. A Gaussian integer π is called a *Gaussian prime* if
- g. The ideal generated by α and β is defined to be $(\alpha, \beta) =$
- h. The Euclidean algorithm for \mathbb{G} states that
- i. If π is a Gaussian prime and $\alpha \in \mathbb{G}$, then $v_\pi(\alpha) =$

2. True/False

a. If α, β are non-zero Gaussian integers, then there exists a unique pair of elements $\kappa, \rho \in \mathbb{G}$ such that $\alpha = \beta\kappa + \rho$ with $N(\rho) < N(\beta)$.

b. If $\gamma \in (\alpha, \beta)$ then $(\alpha, \beta) = (\alpha, \gamma)$.

c. If $N(\alpha) = p$ where p is an odd prime, then $(\alpha, \bar{\alpha}) = \mathbb{G}$.

3. Short Answer

a. Determine a generator for the ideal (α, β) where $\alpha = -1 + 2i$ and $\beta = 6 + 8i$.

- b. For $\alpha = 2 + i$ and $\beta = 3 + 8i$, find Gaussian integers λ, μ such that $\lambda\alpha + \mu\beta = 1$.
- c. For the Gaussian prime $\pi = 2 + i$ and the Gaussian integer $\alpha = 15 - 30i$ compute the π -adic valuation $v_\pi(\alpha)$.
- d. It is a fact that $N(234 + 191i) = 91237$; you don't need to check it. Given this fact, find integers $a, b \in \mathbb{Z}$ such that $a^2 + b^2 = 912370$.

4. Proofs

- a. Prove that the only numbers of norm 2 in \mathbb{G} are $\pm 1 \pm i$, i.e.

$$\{\alpha \in \mathbb{G} \mid N(\alpha) = 2\} = \{1 + i, 1 - i, -1 + i, -1 - i\}.$$

Drawing a picture is okay, but you also need to give the algebraic details.

- b. Prove that if $n = 4k + 3$ with $k \in \mathbb{Z}$, then there is no Gaussian integer α with $N(\alpha) = n$. (p.s. You cannot just quote this from class or homework, you have to prove it. If you quote the result that $n = 4k + 3$ is not the sum of two squares, you'll have to prove that as well.)
- c. Suppose $\alpha, \gamma \in \mathbb{G}$, and $(\gamma) = (\alpha, \alpha + 32)$. Show that $\gamma = \varepsilon(1 + i)^n$ for some $\varepsilon \in \mathbb{G}^\times$ and some integer n in the range $0 \leq n \leq 10$.
- d. Prove that if $n = a^2 + b^2$ and $m = c^2 + d^2$ with $a, b, c, d \in \mathbb{Z}$ then there exist $e, f \in \mathbb{Z}$ such that $mn = e^2 + f^2$.

Extra Credit.

- a. Recall that we did not define $v_\pi(0)$ where π is a Gaussian prime. Is there a reasonable way to define $v_\pi(0)$? Explain.
- b. Prove that there are infinitely many co-prime integer triples (a, b, c) such that $a^2 + b^2 = c^2$. (Such a triple is called a *Pythagorean triple* for reasons you might be able to guess). [A triple (a, b, c) of integers is called co-prime if there is no integer $d > 1$ such that $d|a, d|b$ and $d|c$.]