Proof. See Exercise 2R. □

Corollary 2.39. *If A, B, and C are noncollinear points, then $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$ are all distinct.* □

Theorem 2.40. *If A, B, and C are collinear points and neither B nor C is equal to A, then $\overleftrightarrow{AB} = \overleftrightarrow{AC}$.*

Proof. See Exercise 2S. □

Theorem 2.41. *Given two distinct, nonparallel lines, there exists a unique point that lies on both of them.*

Proof. See Exercise 2T. □

Recall Statements I and II that we introduced in Chapter 2 (see p. 26). As you are asked to show in Exercise 2D, Statement II is independent of the axioms of incidence geometry because it is true in some models and false in others, so it cannot be a theorem of incidence geometry. On the other hand, Statement I *is* a theorem of incidence geometry, as you can now show.

Theorem 2.42. *Given any point, there are at least two distinct lines that contain it.*

Proof. See Exercise 2U. □

Exercises

For Exercises 2A–2J, *you do not need to write out formal proofs. Just answer the questions and* (*if requested*) *give convincing informal explanations of why your answers are correct.*

2A. For each of Examples 2.8, 2.9, and 2.10, figure out which incidence axiom(s) are satisfied and which are not.

2B. Use models to show that each of the incidence axioms is independent of the other three.

2C. Suppose we replace Incidence Axiom 4 with the following:

- **Incidence Axiom 4′:** *Given any line, there are at least three distinct points that lie on it.*

What is the smallest number of points in a model for this geometry? More precisely, find a number n such that every model has at least n points and there is at least one model that has only n points, and explain why your answer is correct.

2D. Use models to show that each of the following statements is independent of the axioms of incidence geometry:

(a) Given any line, there are at least two distinct points that do not lie on it.
(b) Given any point, there are at least three distinct lines that contain it.
(c) Given any two distinct points, there is at least one line that does not contain either of them.

2E. Show that the Fano plane and the seven-point plane are not isomorphic to each other. [Remark: It is not enough just to show that a *particular* correspondence is not an isomorphism; you need to demonstrate that there cannot exist *any* isomorphism between the two models.]

2F. Find a model of incidence geometry that has exactly four points but that is not isomorphic to the four-point plane, and explain why it is not isomorphic.

2G. Determine which parallel postulates, if any, are satisfied by the interpretations of Examples 2.3–2.10.

2H. For each of the following interpretations of incidence geometry, decide if it's a model, and decide which parallel postulate(s) (if any) it satisfies. (No proofs necessary.)

(a) ***Point*** means an ordinary line in three-dimensional space; ***line*** means a plane in three-dimensional space; and ***lies on*** means "is contained in."

(b) ***Point*** means an ordinary line containing the origin in three-dimensional space; ***line*** means an ordinary plane containing the origin in three-dimensional space; ***lies on*** means "is contained in."

2I. Define a model of incidence geometry with ***points*** $1,2,3,4,5,6$; ***lines*** consisting of the sets

$$\{1,2,3\},\ \{3,4,5\},\ \{5,6,1\},\ \{1,4\},\ \{2,5\},\ \{3,6\},\ \{2,6\},\ \{4,6\},\ \{2,4\};$$

and ***lies on*** meaning "is an element of" (Fig. 2.18). (You may use without proof the fact that this is indeed a model.) Which, if any, of the parallel postulates is (are) satisfied by this model? Can you figure out what is remarkable about parallelism in this model?

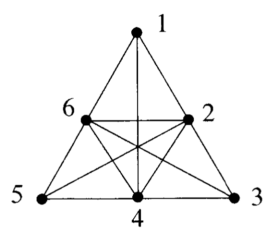

Fig. 2.18. The model of Exercise 2I.

2J. Here is an axiomatic system for an unusual kind of geometry; let's call it ***three-two geometry***. The primitive terms are exactly the same as for incidence geometry: ***point***, ***line***, and ***lies on***. As in incidence geometry, we define ***ℓ contains A*** to mean "A lies on ℓ." There are two axioms:

- **Axiom 1:** *Every line contains exactly three points.*
- **Axiom 2:** *Every point lies on exactly two lines.*

Try to answer as many of the following questions as you can, and justify your answers.

(a) Is this axiom system consistent?

(b) Are the two axioms independent of each other?

(c) What is the minimum number of points in a model for three-two geometry?

(d) Can you find a model with exactly three points?

(e) Can you find a model with exactly three lines?