UMASS AMHERST MATH 300 F. HAJIR

FINAL EXAM REVIEW

Concepts we have learned throughout the course will appear on the final. However, **there will be a decided imbalance toward having more questions from the last few weeks of the course, specifically number theory, countable and uncountable sets, complex numbers)**. Correspondingly, on this sample exam, I have included mostly questions from the last few weeks of the course. HOWEVER, you should review exams 1 and 2 as well as the sample exams to recall the material we discussed earlier in the term.

As with Exams 1 and 2, the final will have three parts: 1. Defintions, 2. Short Answer, and 3. Problems.

Be sure to memorize the definitions so you can move on to the problems section as quickly as possible. You will have 2 hours for completing the final, so this exam will be a little longer than Exams 1 and 2.

Sample Final Exam

1. Definitions

[Review **all** the definitions I asked you on exams 1 and 2 as well as on the sample exams]

a. If a and b are integers, $\gcd(a, b)$ is

b. Two integers a, b are relatively prime if

c. We say $a|b$ if (don't just rephrase, give the actual mathematical property defining this concept)

d. We say $a \equiv b \bmod n$ if

e. State Bézout's theorem.

f. If X is a set equipped with an equivalence relation $\sim$, then the set $\widetilde{X}$ is defined to be

g. A relation $\sim$ from a set X to itself is said to be transitive (h. reflexive; i. symmetric, j. an equivalence relation) if

k. If $\sim$ is a relation from X to Y, and $x \in X$, then the fiber above x is defined by $R_{x,\bullet} =$

l. If $\sim$ is an equivalence relation on X, then a $\sim$-equivalence class is

m. If $x, y \in \mathbb{R}$, the real part (n. imaginary part; o. modulus; p. complex conjugate) of $z = x + iy$ is

q. The triangle inequality states that if $w, z \in \mathbb{C}$, then

r. Cantor's theorem states that

s. A set X is countable means that

t. A set X is uncountable means that

u. If X is a set, $|X| = \aleph_0$ means that

v. If X is a set, $|X| = \aleph_1$ means that

w. If X and Y are arbitrary sets, $|X| = |Y|$ means that

x. If X and Y are arbitrary sets, $|X| \leq |Y|$ means that

2. Short Answers

a. The quantity $\min\{x \in \mathbb{Z} \mid a|x \wedge b|x\}$ is called

b. In the above sentence, this minimum is guaranteed to exist by the .... Principle because ....

c. Use the Euclidean algorithm to compute $\gcd(432, 168)$ as well as $\operatorname{lcm}(432, 168)$.

d. Sketch and shade the region in the complex plane defined by

$$R = \{z \in \mathbb{C} \mid |z-1| < |z+1|\}.$$

e. For $w \in \mathbb{C}$, define a map $\delta_w : \mathbb{C} \to \mathbb{C}$ via $\delta_w(z) = wz$ for all $z \in \mathbb{C}$. If $w = 1 + i\sqrt{3}$, the map δ_w can be represented by a radial dilation by a factor .... followed by a counterclockwise rotation around the origin of measure ....

f. Suppose z, w, v are three complex numbers such that $|z - w| = |z - v| + |v - w|$. Draw a picture of what this means geometrically. What can you conclude about the geometric configuration of the complex numbers z, w, v?

g. Write the complex number $z = (7 + 4i)/(3 - 2i)$ in polar form, i.e. find real numbers r, θ such that $z = re^{i\theta}$.

h. TRUE or FALSE: If $z_0 \in \mathbb{C}$, then the equation $w^5 = z_0$ has 5 distinct solutions in $\mathbb{C}$.

i. TRUE or FALSE: If X is a countable set, and Δ is a partition of X, then Δ is also a countable set.

j. TRUE or FALSE: If $x, y \in \mathbb{Z}$, and $3x + 17y = 2$, then $\gcd(x, y)$ is either 1 or 2.

k. TRUE or FALSE: The equation $3x + 18y = 1$ has no solution with $x, y \in \mathbb{Q}$.

l. Write down two **uncountable** sets, X and Y, which are not equivalent to each other.

m. Give a bijection $(0, 1) \to \mathbb{R}$.

n. Consider the set $X = \{1, 2, 3, 4, 5\}$ and the map $f : X \to \mathcal{P}(X)$ defined by $f(1) = \{4\}$, $f(2) = \{3, 4\}$, $f(3) = \{2, 3, 4\}$, and $f(4) = \{1, 2, 3, 4\}$. Calculate $Y_f = \{x \in X \mid x \notin f(x)\}$. Is Y_f in the image of f? Are you surprised by this? Why or why not?

o. Write the number 0.125 in base 5.

TRUE OR FALSE: For each statement below, Indicate whether it is True or False.
Every infinite subset of an uncountable set is uncountable.
For *arbitrary* sets X, Y, Z, if $|X| \leq |Y|$ and $|Y| \leq |Z|$, then $|X| \leq |Z|$.
If X is an infinite set, then there is a injection $X \to \mathbb{N}$.
If $|X| = \aleph_0$, then $|X \times X| = \aleph_0$.
If X is equivalent to Y, then there is an injection $X \hookrightarrow Y$ as well as an injection $Y \hookrightarrow X$.
If X is a countable set, then every infinite subset of X is equivalent to X.

3. Problems

A. Prove Cantor's theorem: If X is an arbitary set, and $f : X \to \mathcal{P}(X)$ is a map from X to the set of all subsets of X, then f is not surjective. Hint: Proof by contradiction.

B. Suppose $r, s, m, n \in \mathbb{Z}$ and $\gcd(m, n) = 1$.

(i) Show that the set

$$\{x \in \mathbb{Z} \mid x \equiv r \bmod m \wedge x \equiv s \bmod n \wedge 0 \leq x \leq mn - 1\}$$

is a singleton. In other words, there exists a unique integer in the interval $[1, mn]$ that gives remainder r when divided by m and remainder s when divided by n.

Hint: By Bézout, we can find a, b such that $am + bn = 1$. Now try $x = ams + bnr$. It satisfies two of the three needed conditions. How do you "fix" it to get the third condition?

(ii) How many integers in the interval $[0, 5000]$ give remainder 73 when divided by 100 and remainder 1 when divided by 37? What is the least such integer?

C. Let p be a prime number. Let $X = \mathbb{Z}$ be the set of integers, and for $x, y \in \mathbb{Z}$, write $x \sim y$ if and only if $x \equiv y \bmod p$, i.e. if and only if $p|(x-y)$. Thus, the set $\widetilde{X}$ has p elements, namely $\widetilde{0}, \widetilde{1}, \ldots, \widetilde{p-1}$. On the set $\widetilde{X}$, let us define two operations, $+, \times$ as follows:

$$\widetilde{a} + \widetilde{b} := \widetilde{a+b}, \qquad \widetilde{a}\widetilde{b} := \widetilde{ab}.$$

Show that if $\widetilde{a} \neq \widetilde{0}$, then there exists $b \in \mathbb{Z}$ such that $\widetilde{a}\widetilde{b} = \widetilde{1}$.

D. Describe a bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, thereby proving that $\mathbb{N}$ is countable. (Drawing a picture is a good idea, but it should be accompanied by a careful description of the map you are constructing).

E. State the triangle inequality, then use it to prove that for $u, v \in \mathbb{C}$,

$$|u| - |v| \leq |u - v|.$$

F. Find six complex roots of the equation $z^6 + z^3 + 1 = 0$. Hint: let $w = z^3$ so that $w^2 + w + 1 = 0$. Solve for w and put the solutions w_1, w_2 in $re^{i\theta}$ form, then solve $z^3 = w_1$ and $z^3 = w_2$.

G. (i) Suppose $z_0, z_1 \in \mathbb{C}$ and $z_0 \neq z_1$. Consider a map $z : [0,1] \to \mathbb{C}$ defined by $z(t) = tz_1 + (1-t)z_0$ for $0 \leq t \leq 1$. Note that $z(0) = z_0$ and $z(1) = z_1$. Thinking of this map as a path in the complex plane, describe (geometrically) what this path is.

(ii) Let $B = \{z \in \mathbb{C} \mid |z| < 1\}$ be the inside of the unit circle; it's called the unit disc. Use the triangle inequality to show that, given two distinct points $z_0, z_1 \in B$, every point of the line segment joining z_0 to z_1 is inside B also. (This is clear geometrically, I am asking for an "algebraic" proof). You have just shown that the unit disc is *convex.*

H. Use strong induction to prove that if $n \geq 2$ is an integer, then there exists a prime number p such that $p|n$. (Do not use the fact that integers factor into products of prime powers).

4. Extra Credit

A. Suppose $n \geq 2$ is an integer. Write down an explicit formula for a non-identity 2 by 2 matrix M with real entries such that $M^n = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.