UMASS AMHERST MATH 300 SP '09, F. HAJIR

EXAM 2 REVIEW

Exam 2 will cover the material introduced in Homeworks, 1-7, with the emphasis being heavily on HW 5, 6, and 7, i.e. on equivalence relations, partitions, induction (including the binomial theorem), and divisibility. Be sure to be familiar with the equivalence relation of congruence modulo n, called $\sim_n$ and also $\equiv \text{mod} n$, on $\mathbb{Z}$ (for an arbitrary positive integer n) defined by $x \sim_n y$ (also written $x \equiv y \bmod n$) if and only if $n|(x-y)$.

As with Exam 1, Exam 2 will have three parts: 1. Defintions, 2. Short Answer, and 3. Problems. The problems will usually but not always require you to write a cogent, concise and correct proof. Some of the problems will be statements that were already proved in class or are taken directly from homework. But at least some of the problems will require you to prove a statement that has not been presented to you before, or to find a counterexample to a statement.

For the definitions, it is important to be extremely precise. Be sure to memorize the definitions well enough so that you can just rattle them off. Many of you lost a large number of points on the Definitions on Exam 1. Also, many of you spent too much time on the Short Answers, which do not count as much as the Problems, then ran out of time.

The points will be distributed approximately as follows: 25% Definitions, 25% Short Answer, and 50% Problems.

Here begineth the sample exam.

1. Definitions

The Well-ordering Principle states that

The Principle of Mathematical Induction states that

The Principle of Complete Mathematical Induction (or Strong Induction) states that

The Binomial Theorem states that

A partition of a set X is

A relation R from X to Y is a set is

The graph of a relation R is

An equivalence relation on a set X is

If R is a relation on X and $x \in X$, then the fiber $R_{\bullet,x}$ is defined to be

If a and b are integers, $a|b$ means that

An integer n is prime if

For integers a, b, $\gcd(a, b)$ is defined to be

2. Short Answer

Suppose $X = \mathbb{Z}$ and define a relation $\sim$ on X by $x \sim y$ if and only if $x + y$ is even. Show briefly that this is an equivalence relation. Describe the associated partition of X. Describe the set $\tilde{X} = X/\sim$, and the map $X \to \tilde{X}$.

Let $X = \mathbb{R}$ be the set of real numbers, and define a map $f : X \to \{-1, 0, 1\}$ by $f(x) = |x|/x$ if $x \neq 0$ and $f(0) = 0$. Associated to this map, there is a partition of $\mathbb{R}$. Describe this partition; how many elements does it have? Define the equivalence relation on $\mathbb{R}$ associated to this map f.

Is $\Delta = \{\{\}, \{1\}, \{2\}\}$ a partition of $X = \{1, 2\}$? Why or why not?

List all partitions of $X = \{1, 2, 3\}$. How many distinct equivalence relations on X are there?

Suppose $a_0, a_1, a_2, \ldots$ is a sequence of integers. Is it true that if a_0 and a_1 are even integers and $a_{n+1} = 17a_n + 2005a_{n-1}$ for all $n \geq 3$, then a_n is even for all n? Why or why not?

Suppose $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is equipped with the partition

$$\Delta = \{\{1, 2\}, \{3\}, \{4, 5, 6\}, \{7, 8, 9, 10\}\}.$$

If $\sim$ is the equivalence relation corresponding to Δ, then what is the cardinality of the set $X/\sim$?

Use the extended Euclidean algorithm to compute $d = \gcd(873, 132)$ and find integers r, s such that $873r + 132s = d$. Also compute $\operatorname{lcm}(873, 132)$.

3. Problems

Define a sequence of integers g_n for $n \geq 1$ by the following recursion rule:

$$g_1 = 1, \qquad g_{n+1} = g_n + n - 1 \qquad \text{for all } n \geq 1.$$

Use the principle of mathematical induction to prove that $g_n = n(n+1)/2$ for all integers $n \geq 1$.

Use the principle of mathematical induction to prove that $3 | 2^{2n} - 1$ for all $n \geq 1$.

Use the principle of mathematical induction to prove that for all integers $n \geq 1$,

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Use complete induction to prove that if x a real number with the property that $x + 1/x \in \mathbb{Z}$, then $x^n + 1/x^n \in \mathbb{Z}$ for all $n \geq 1$.

Prove that $F_n < (7/4)^{n-1}$ for all $n \geq 1$, where F_n is the nth Fibonacci number: $F_0 = F_1 = 1, F_{n+1} = F_n + F_{n-1}, n \geq 2$.

For integers, a, b, c, we define $\gcd(a, b, c) = \operatorname{Div}^+(a) \cap \operatorname{Div}^+(b) \cap \operatorname{Div}^+(c)$. Show that $\gcd(a, b, c) = \gcd(a, \gcd(b, c))$.

Suppose X is set and $\sim$ and $\approx$ are equivalence relations on X. Suppose for all $x, y \in X$, $x \sim y \Rightarrow x \approx y$. Show that there is a surjective map $X/\approx \to X/\sim$.

Let $X = \mathbb{R}^2 \setminus \{(0, 0)\}$ be the plane punctured at the origin and define a relation $\sim$ on X by $(x, y) \sim (x', y')$ if and only if there exists $t \in \mathbb{R} \setminus \{0\}$ such that $x' = tx$ and $y' = ty$. Give a geometric interpretation of this relation. Show that it is an equivalence relation. Let $Y = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle and define a map $X/\sim \to Y$ which is an isomorphism. (Use the geometric interpretation).

Suppose $f : X \to Y$ is a map, and define a relation $\sim$ on X by $x \sim y$ if and only if $f(x) = f(y)$. Prove that $\sim$ is an equivalence relation on X.