UMASS AMHERST MATH 300 SP '05, F. HAJIR

HOMEWORK 6 SOLUTIONS

1. Problems

1. Prove by induction that, for all integers $n \geq 1$,

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1.$$

Base Case: if $n = 1$, $\sum_{i=0}^{0} 2^i = 2^0 = 1 = 2^1 - 1$. That works.
Inductive Step: Suppose $\sum_{i=0}^{k-1} 2^i = 2^k - 1$ for some $k \geq 1$. Then

$$\begin{aligned} \sum_{i=0}^{k} 2^i &= \left(\sum_{i=0}^{k-1} 2^i\right) + 2^k \\ &= 2^k - 1 + 2^k \\ &= 2 \cdot 2^k - 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

which is what we wanted to prove. Thus, by PMI, we are done.

2. Prove by induction that for $n \geq 1$, the sum of the cubes of the first n positive integers is the square of the sum of the first n positive integers, i.e.

$$\sum_{j=1}^{n} j^3 = \left(\sum_{i=1}^{n} i\right)^2.$$

Feel free to use the formula for Bowling Numbers that we proved in class and in these notes.

We recall that $B_n = 1 + 2 + \cdots + n = n(n+1)/2$, so we wish to prove $P(n)$ for all $n \geq 1$ where $P(n)$ is the statement that

$$\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Base case: $n = 1$ gives $1^3 = (1(2)/2)^2$. That works.

Inductive Step: Suppose for some $k \geq 1$, $\sum_{j=1}^{k} j^3 = (k(k+1)/2)^2$. Then

$$\begin{aligned}
\sum_{j=1}^{k+1} j^3 &= (k+1)^3 + \sum_{j=1}^{k} j^3 \\
&= (k+1)^3 + (k(k+1)/2)^2 \\
&= (k+1)^2((k+1) + k^2/4) \\
&= (k+1)^2(k^2+4k+4)/4 \\
&= (k+1)^2(k+2)^2/4.
\end{aligned}$$

Since $B_{k+1} = (k+1)(k+2)/2$, we have shown that $P(k)$ implies $P(k+1)$.
Thus, by PMI, we are done.

3. Prove by induction that, for all integers $n \geq 1$,

$$\frac{1}{n!} \leq \frac{1}{2^{n-1}}.$$

We let $P(n)$ be the statement $1/n! \leq 1/2^{n-1}$.
Base case: $n = 1$ gives $1/1! \leq 1/2^0$ which is true.
Inductive case: Suppose $1/k! \leq 1/2^{k-1}$ for some integer $k \geq 1$. Since $k \geq 1$, we have $k+1 \geq 2$, hence $1/(k+1) \leq 1/2$. Then,

$$\begin{aligned}
\frac{1}{(k+1)!} &= \frac{1}{k!} \cdot \frac{1}{k+1} \\
&\leq \frac{1}{2^{k-1}} \cdot \frac{1}{k+1} \\
&\leq \frac{1}{2^{k-1}} \frac{1}{2} \\
&\leq \frac{1}{2^k}.
\end{aligned}$$

Thus, we have shown that $P(k) \Rightarrow P(k+1)$ for $k \geq 1$.
By PMI, we are done.

4. Prove, using induction, that if X is a finite set of size $n \geq 0$, then X has 2^n subsets, i.e. that $|\mathcal{P}(X)| = 2^{|X|}$. You gave a proof of this on HW5. Your proof here must use induction on the size of X.

Base Case: $n = 0$. Let X be a finite set of size 0. Then $X = \{\}$ is the empty set, which has only one subset, namely itself. Thus, $|P(X)| = 1 = 2^0 = 2^{|X|}$ in this case, as desired.

Inductive Step. Let $k \geq 0$ be an integer, and suppose that for all finite sets X of size k, $|\mathcal{P}(X)| = 2^k$. Now, let Y be a set of size $k+1$. We must show that $|\mathcal{P}(Y)| = 2^{k+1}$. Since $|Y| \geq 1$, Y is not the empty set. So we may choose y to be a fixed element of Y. Then, we let $X = Y \setminus \{y\}$ be the complement of $\{y\}$ in Y. It is clear that $|X| = k$, hence by the induction hypothesis, $|\mathcal{P}(X)| = 2^k$. Now let $\mathcal{P}_0(Y)$ be the subset of $\mathcal{P}(Y)$ consisting of those subsets of Y which do not contain y and let $\mathcal{P}_1(Y)$ be the subset of $\mathcal{P}(Y)$ consisting of subsets of Y which do contain y. Since every subset of Y either contains y or does not contain y, $\{\mathcal{P}_0(Y), \mathcal{P}_1(Y)\}$

is a partition of $\mathcal{P}(Y)$, so $|\mathcal{P}(Y)| = |\mathcal{P}_0(Y)| + |\mathcal{P}_1(Y)|$. It is easy to see, on the other hand, that the sets $\mathcal{P}(X)$, $\mathcal{P}_0(Y)$ and $\mathcal{P}_1(Y)$ are all bijective to each other. First of all $\mathcal{P}(X)$ and $\mathcal{P}_0(Y)$ are simply equal! Second, the maps $f : \mathcal{P}_0(Y) \to \mathcal{P}_1(Y)$ and $g : \mathcal{P}_1(Y) \to \mathcal{P}_0(Y)$ defined by $f(S) = S \cup \{y\}$ and $g(T) = T \setminus \{y\}$ are clearly well-defined and inverses of each other, hence f and g are bijective, so in particular $|\mathcal{P}_1(X)| = |\mathcal{P}_0(X)| = |\mathcal{P}(X)| = 2^k$. Putting all this together, we have $|\mathcal{P}(Y)| = 2^k + 2^k = 2^{k+1}$, which what we wanted to prove. We are done by PMI.

5. Prove by induction that for $n \geq 2$,

$$\prod_{j=2}^{n}(1 - \frac{1}{j^2}) = \frac{n+1}{2n},$$

Base Case: $n = 2$. $\prod_{j=2}^{2}(1 - 1/j^2) = (1 - 1/2^2) = 3/4 = (2 + 1)/(2 \cdot 2)$ that works.
Inductive step. Suppose for some $k \geq 1$ that $\prod_{j=2}^{k}(1 - \frac{1}{j^2}) = (k + 1)/(2k)$. Then

$$\begin{aligned}\prod_{j=2}^{k+1}(1 - \frac{1}{j^2}) &= \frac{k+1}{2k}(1 - \frac{1}{(k+1)^2}) \\ &= \frac{k+1}{2k}\frac{(k+1)^2 - 1}{(k+1)^2} \\ &= \frac{k^2 + 2k + 1 - 1}{2k(k+1)} \\ &= \frac{k+2}{2(k+1)},\end{aligned}$$

which is what we wished to prove. By PMI, we go home.

6. Prove by induction that if $x > 1$ is a real number then for all integers $n \geq 2$, $(1+x)^n > 1 + nx$.

Base case: $n = 2$. $(1 + x)^2 = 1 + 2x + x^2 > 1 + 2x$ because $x > 1$ imples that $x^2 > 1 > 0$.
Inductive step: Suppose that $(1 + x)^k > 1 + kx$ for some integer $k \geq 2$. Then

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k(1+x) \\ &> (1+kx)(1+x) \\ &\geq 1 + x + kx + kx^2 \\ &\geq 1 + (k+1)x + kx^2 \\ &> 1 + (k+1)x,\end{aligned}$$

the last step being due to the fact that $kx^2 > 0$. We have completed the inductive step, so we are done by PMI.

7. Here is a "proof," using induction, of the statement that all horses are have the same color.

We must prove that for $n \geq 1$,

$H(n)$: If C is a collection of n horses, all the n horses in C have the same color.

We will use induction on n to prove $H(n)$. Base case: $n = 1$; since there is only one horse, it has the same color as all the horses present in the collection, i.e. itself. Inductive Step: we must show that $H(k)$ implies $H(k + 1)$. So we assume known that in any collection of k horses, they all have the same color [the "induction hypothesis"] and must show this to be the case for any collection of $k + 1$ horses. Let C be a collection of $k + 1$ horses. Take any subcollection $S = C \setminus \{h\}$ of size k, i.e. one which leaves out one horse, lets us call it h. We know that all the horses in S have the same color by the induction hypothesis; it remains only to show that h has the same color as all the horses in S. Now let $S' = C \setminus \{h'\}$ be a different collection of k horses in C, i.e. $h' \neq h$. Then all the horses in S' have the same color, again by the induction hypothesis. Since $h \in S'$, h must have the same color as all the other horses too, so all the horses in C have the same color. By the Principle of Mathematical Induction, all horses have the same color.

Write a carefully written discussion of this proof, describing where and how it goes wrong. [For you agree that the statement is false, right?!]

It is correctly proven above that $P(k) \Rightarrow P(k+1)$ for all $k \geq 2$. However, the trouble with this proof is that it does not work for $k = 1$! In the part of the argument where one looks at the set $S' = C \setminus \{h'\}$, where $h' \neq h$, when $k = 1$, no such h' exists because C has size 1! Thus, the logical chain $P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \cdots$ is broken at the very first junction, invalidating this "proof" by induction.

8. Define the famed *Fibonacci Sequence* $F_0, F_1, F_2, \ldots$ as follows. Let $F_0 = 0$, $F_1 = 1$ and define the rest *recursively* by letting $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$. Thus, $F_2 = F_1 + F_0 = 1$, and $F_3 = F_2 + F_1 = 1 + 1 = 2$ and so on, each new term being the sum of the two preceding ones.

Let M be the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(a) Verify for a few small n, say $1 \leq n \leq 5$, that $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$. Try to prove this holds for all n, either using induction, or some other trick. [*Do Try it*, it's good for you, but if it gets hairy, don't sweat it too much, you'll see why in a minute.]

The first few identities are easily verified. I hope you discovered that it was rather non-obvious *how* to apply induction correctly here. That was the point of part (a)!

(b) Using induction on n, prove that, for $n \geq 1$,

$$M^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

[For a reminder about matrix multiplication, see Remembrances of Things Past].

For integers $k \geq 1$, let

$$N_k = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

Base case: $n = 1$ holds clearly because $F_0 = 0$, $F_1 = 1$ and $F_2 = 1$.

Inductive step: Now suppose for some integer $k \geq 1$, that $M^k = N_k$. Then $M^{k+1} = M^k M = N_k M$ and by multiplying these matrices, we find

$$\begin{aligned} M^{k+1} = N_k M &= \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} F_k & F_{k-1} + F_k \\ F_{k+1} & F_k + F_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} F_k & F_{k+1} \\ F_{k+1} & F_{k+2} \end{pmatrix} \end{aligned}$$

where in the very last line we have used the recursive definition of the Fibonacci sequence twice to achieve what we wanted to prove. By PMI, we are done.

(c) Now Use (b) to prove the formula in (a).

Now we take determinants of both sides of the matrix formula in (a). Since $\det(M^n) = \det(M)^n$ [can prove this by induction!] and since $\det(M) = (-1)$, we find $\det(M^n) = (-1)^n = F_{n-1}F_{n+1} - F_n^2$. Now that was CAKE!

9. Suppose $m \geq 2$ and $X_1, \dots, X_m$ are countable sets. Use induction on m to prove that $X_1 \cup X_2 \cup \cdots \cup X_m$ is countable.

We have a theorem to the effect that if X and Y are countable then $X \cup Y$ is countable. The proof is just to interleave the elements. That takes care of the base case $m = 2$. Now suppose $k \geq 2$ is an integer such that whenever we have k countable sets, their union is countable. Suppose $X_1, \dots, X_k, X_{k+1}$ are $k+1$ countable sets. Then

$$X_1 \cup X_2 \cup \cdots \cup X_{k+1} = X \cup X_{k+1}$$

where $X = X_1 \cup \cdots \cup X_k$. By the induction hypothesis, X is countable since it's the union of k countable subsets, and by using the base case theorem, the union of two countable sets, such as $X \cup X_{k+1}$ is countable, so $X_1 \cup \cdots \cup X_{k+1} = X \cup X_{k+1}$ is countable. That takes care of the induction step, so by PMI, we are done.

10. Use induction to prove that for any integer $n \geq 1$, the number $a_n = 5^n + 2(3^{n-1}) + 1$ is divisible by 8, i.e. $a_n/8$ is an integer.

We check $n = 1$: $a_1 = 5 + 2(3^0) + 1 = 5 + 2 + 1 = 8$ is divisible by 8.

Suppose for some $k \geq 1$, a_k is divisible by 8. So we can write $a_k = 8t$ for some integer t. Then

$$\begin{aligned} a_{k+1} - a_k &= 5^{k+1} + 2(3^k) + 1 - (5^k + 2(3^{k-1}) + 1) \\ &= 5^k(5-1) + 2(3^{k-1})(3-1) \\ &= 4(5^k) + 4(3^{k-1}) \\ &= = 4(5^k + 3^{k-1}) \\ &= 4(\text{odd} + \text{odd}) \\ &= 4(\text{even}) \end{aligned}$$

which shows that $a_{k+1} - a_k = 8s$ for some integer s. Thus, by the induction hypothesis, $a_{k+1} = 8t + 8s$ is divisible by 8. By PMI, we are done.