UMASS AMHERST MATH 300: HW 7

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Countable and uncoutable sets

1. (a) For sets X, Y, we write $X \sim Y$ if there exists a bijection from X to Y. Recall that for each $n \in \mathbb{N}$, $\mathbb{P}_n = \{1, 2, \ldots, n\}$; $\mathbb{P}_0 = \{\}$ is the empty set. Recall also that a set X is called finite if there exists $n \in \mathbb{N}$ such that $X \sim \mathbb{P}_n$, in which case we say that X has cardinality n (or order n). Suppose X and Y are two finite sets of cardinality n. Show that $X \sim Y$, i.e. show that there is a bijection from X to Y.
(b) Suppose X and Y are two infinite countable (also called "countably infinite") sets. Prove that $X \sim Y$.

2. Prove that $\mathbb{N} \times \mathbb{N}$ is countable.

3. Show that if X is a countable set, and $Y \subseteq X$, then Y is countable.

4. Give a bijection from $(0, 1) = \{x \in \mathbb{R} | 0 < x < 1\}$ to $\mathbb{R}$, thereby showing that $|(0, 1)| = |\mathbb{R}|$. Hint: think about a function that has an asymptote going to $-\infty$ near 0 and one going to $+\infty$ near 1.

5. (a) Show that if X and Y are countable sets, then $X \cup Y$ is a countable set. (Hint: if X and Y are both countably infinite, say $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$, then *interleave the two sequences* (the way the odds and evens are interleaved)).
(b) Let $I = \mathbb{R} \setminus \mathbb{Q}$ be the set of irrational numbers. Prove that I is uncountable. (Hint: Proof by contradiction is your friend).

6. Suppose X is a non-empty set and $f : X \to \mathcal{P}(X)$ is defined $f(x) = X \setminus x$. Consider the subset $Y_f = \{x \in X | x \notin f(x)\}$ of X (which plays a prominent role in Cantor's theorem). Determine Y_f for the particular f we have just defined.

7. (a) Convert the rational number 147.05 (written in base ten) to base 4.
(b) Convert the base 3 rational number $(120.\overline{21})_3 = (120.21212121\cdots)_3$ to base ten.

Complex Numbers

9. (There are no zero-divisors in $\mathbb{C}$). Show that if $z, w \in \mathbb{C}$, and $zw = 0$ then either $z = 0$ or $w = 0$. (you may use the fact that this is true for $z, w \in \mathbb{R}$).

10. (a) (Every non-zero complex number is invertible). Show that for each $z \in \mathbb{C}$ such that $z \neq 0$, there exists a unique $w \in \mathbb{C}$ such that $wz = 1$, so it's okay to write $w = z^{-1} = 1/z$.
(b) Use (a) to give another proof of the statement in Problem 9.
(c) For $z = 3 + 4i$, determine $1/z$ and write it in the form $a + bi$ with real numbers a, b.

11. (a) Show that for $z \in \mathbb{C}$, $z = 0$ if and only if $|z| = 0$.
(b) Prove that $|zw| = |z||w|$.
(c) Prove using induction that for all $n \in \mathbb{Z}$, $|z^n| = |z|^n$.

12. (a) Show that for $z, w \in \mathbb{C}$, $|z - w|$ is the usual distance from z to w.
(b) (Triangle Inequality) Give an algebraic proof of the fact that for $z, w \in \mathbb{C}$, $|z - w| \leq |z| + |w|$ and interpret this fact geometrically. Hint: First prove that if $u \in \mathbb{C}$, then $\Re(u) \leq |u|$. Next, argue that it suffices to show that $|z - w|^2 \leq (|z| + |w|)^2$. Now justify each step in the following:

$$|z - w|^2 = (z - w)(\overline{z} - \overline{w}) = |z|^2 + |w|^2 + 2\Re(-z\overline{w}) \leq |z|^2 + |w|^2 + 2|z\overline{w}| = (|z| + |w|)^2.$$

(c) Shade in the region $\{z \in \mathbb{C} \mid 1 \leq |z - i| \leq 2\}$. It is called an "annulus." Hint: $|z - i|$ is the distance from z to i.

13. (a) Find four solutions in $\mathbb{C}$ of the equation $z^4 = 1$.
(b) Using your vast knowledge of trigonometry, evaluate $\zeta = \cos(\theta) + i\sin(\theta)$ where $\theta = 2\pi/6$.
(c) Verify that $1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5$ are six distinct solutions of $z^6 = 1$. They are called the sixth roots of unity in $\mathbb{C}$.
(d) Draw a fairly accurate picture of the unit circle showing that the roots of $z^4 = 1$ and $z^6 = 1$ all lie on it. (Label the solutions). Use red for the 4 solutions of one equation and Blue for the six solutions of the other.

14. (Autour le théorème de De Moivre) For $z = r(\cos(\theta) + i\sin(\theta)) \in \mathbb{C}$, prove using induction on n that for all $n \in \mathbb{Z}$, $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

Extra Credit Problems.

1. Prove that the points z_1, z_2, z_3 in the complex plane are vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3.$$

2. Let $\zeta = e^{2\pi i/5}$ so that $1, \zeta, \zeta^2, \zeta^3, \zeta^4$ are the vertices of a regular pentagon. The diagonals of this pentagon meet at the vertices of a smaller regular pentagon. Determine them.

3. (a) Show that for $A \neq 0$, the set of all points (x, y) in $\mathbb{R}^2$ satisfying $Ax^2 + Ay^2 + Bx + Cy + D = 0$ is either empty or a circle. Determine the center and the radius. What happens when $A = 0$?

(b) Suppose $z_1, z_2 \in \mathbb{C}$ are distinct fixed points in $\mathbb{C}$ and K is a fixed positive real number, $K \neq 1$. Show that the set of all $z \in \mathbb{C}$ satisfying

$$\frac{|z - z_1|}{|z - z_2|} = K$$

is a circle. Where is its center? What is its radius? How are z_1, z_2 positioned vis à vis this circle? If we keep K fixed and move z_1 along a straight line toward z_2, what happens to the center and radius of the circle? What happens when we move z_1 along the same straight line away from z_2? If we keep $z_1, z2$ fixed and move K toward 0 or toward ∞, what happens to the circle? What happens when $K = 1$?

4. (a) Let S be a set of size $n \geq 1$ and suppose r is an integer in the range $0 \leq r \leq n$. Let

$$\mathbb{P}_r(S) = \{T \subseteq S \mid |T| = r\}$$

be the set of all subsets of S of cardinality r. Use the multiplication counting principle to deduce that

$$|\mathbb{P}_r(S)| = \frac{n!}{r!(n-r)!}.$$

This number is often denoted by $\binom{n}{r}$.

(b) With the above notations for n and r and for variables x and y, derive the binomial formula

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

5. (a) Use the well-ordering principle to prove the Principle of Double Induction: Suppose for each pair $(a, b) \in \mathbb{N} \times \mathbb{N}$, we have a statement $P(a, b)$. Suppose i) $P(1, 1)$ is true, and ii) Whenever $P(k, l)$ true for some $(k, l) \in \mathbb{N} \times \mathbb{N}$, then $P(k + 1, l)$ and $P(k, l + 1)$ are also true. Then $P(a, b)$ is true for all $(a, b) \in \mathbb{N}$.

(b) Now prove a slight modification: Suppose for all integers $n, r \geq 1$ with $r \leq n$, we have a statement $P(n, r)$. Suppose i) $P(1, 1)$ is true and ii) Whenever $P(k, l)$ is true for some $(k, l) \in \mathbb{N} \times \mathbb{N}$ with $l \leq k$, then $P(k + 1, l)$ and $P(k + 1, l + 1)$ are true. Then $P(a, b)$ is true for all $(n, r) \in \mathbb{N}$ with $r \leq n$.

6. For a positive integer n, we let $I_n = \{k \in \mathbb{Z} \mid 1 \leq k \leq n\}$ be the set of integers from 1 to n. If T is a subset of I_n, let m_T be the least element of T. For $1 \leq r \leq n$, let $f(n, r)$ be the average, over all subsets T of I_n of cardinality r, of m_T. Recalling from problem 4 above that there are $\binom{n}{r}$ subsets of cardinality r in I_n, we have, therefore,

$$f(n, r) := \frac{1}{\binom{n}{r}} \sum_{T \subseteq I_n, |T| = r} m_T.$$

Prove that

$$f(n, r) = \frac{n + 1}{r + 1}.$$