UMASS AMHERST MATH 300 FALL 05 F. HAJIR

HW 6

1. Reading

You should read Part 7 in my online notes as well as Chapter 2 of Gilbert/Vanstone

2. Problems from Gilbert/Vanstone

Exercise Set 2: 11,18,27,30,36
Problem Set 2: 73

3. Problems from Farshid's Brain

1. Suppose $a, b, c \in \mathbb{Z}$.
(a) Show that if $a|b$ and $c \neq 0$, then $ca|cb$.
(b) Show that if $a|b$ and $b|c$, then $a|c$.
(c) Show that if $a|b$ and $a|c$, then $a|(mb + nc)$ for all $m, n \in \mathbb{Z}$.

2. Show that there are arbitrarily long sequences of consecutive integers containing no primes. In other words, show that given an integer $N \geq 1$, there exists an integer a such that $a+1, a+2, \ldots, a+N$ are all composites. Hint: try $a = N! + 1$. Look for an "obvious" divisor of $a+1$, an "obvious" divisor of $a+2$ etc.

3. Suppose a, b, n are integers, $n \geq 1$ and $a = nd + r$, $b = ne + s$ with $0 \leq r, s < n$, so that r, s are the remainders for $a \div n$ and $b \div n$, respectively. Show that $r = s$ if and only if $n|(a - b)$. [In other words, two integers give the same remainder when divided by n if and only if their difference is divisible by n.]

4. If $n \geq 1$ and $m_1, \cdots, m_n \in \mathbb{Z}$ are n integers whose product is divisibe by p, then at least one of these integers is divisible by p, i.e. $p|m_1 \cdots m_n$ implies that then there exists $1 \leq j \leq n$ such that $p|m_j$. Hint: use induction on n.

5. (a) Calculate $\gcd(315, 168)$ using the Euclidean algorithm, then use this information to calculate $\mathrm{lcm}(315, 168)$. Determine integers x, y such that $315x + 168y = \gcd(315, 168)$. You may use the Blankinship version of the Bezout algorithm if you wish. Now obtain the prime factorizations of 315 and 168 to double-check your computation of the gcd and lcm of 315 and 168.
(b) Calculate $\gcd(89, 148)$ using the Euclidean algorithm.

6. (a) Show that if $n > 1$ is composite, then there exists d in the range $1 < d \leq \sqrt{n}$ such that $d|n$. (Hint: you might want to use proof by contradiction).

(b) Use (a) to show that if n is not divisible by any integers in the range $[2, \sqrt{n}]$, then n is prime.

(c) Use (b) to show that if n is not divisible by any **primes** in the range $[2, \sqrt{n}]$, then n is prime.

(d) Use the procedure in (c) to verify that 229 is prime.

(e) Suppose you write down all the primes from 2 to n. We know that 2 is a prime so we circle it and cross out all other multiples of 2. The next uncrossed number is 3 and we claim that 3 therefore must be prime. Explain why. Now cross out all the multiples of 3. The next uncrossed number is 5 so we claim it must be a prime. We continue in this fashion until we get to $\sqrt{n}$. Explain why all the remaining numbers are prime. Carry out this procedure for $n = 100$ to find all the primes less than 100. This is called the Eratosthenes sieve. (You may want to write them in 10 rows of 10 numbers each).

7. Prove that if $n \in \mathbb{N}$, then $\gcd(n, n+1) = 1$.

8. Suppose x is a real number such that $x + 1/x$ is an integer. Show that $x^n + 1/x^n$ is also an integer for all $n \geq 1$. (Hint: Use complete induction on n).

9. Here is a "proof" by complete induction that all Fibonacci numbers are even! Your job is to explain the error in the argument.

For $n \geq 0$, let $P(n)$ be the statement that F_n is even. We will prove $P(n)$ by complete induction on n. We check the base case, $P(0)$: $F_0 = 0$ is even. Now we move to the induction step: We must show that if $P(j)$ holds for $0 \leq j \leq n$, then $P(n)$ holds. Well, if $P(j)$ holds for $0 \leq j \leq n$, then $F_{n+1} = F_{n-1} + F_n$ is even because F_{n-1} and F_n are even by $P(n-1)$ and $P(n)$, respectively. By Complete Induction, therefore, F_n is even for all $n \geq 0$.

10. Show that for $n \geq 2$, in any set of $2^n - 1$ integers, there is a subset of exactly 2^{n-1} of them whose sum is divisible by 2^{n-1}. (Hint: use ordinary induction on n; assuming you can do it for any set of size $2^k - 1$, suppose you have a set of size $2^{k+1} - 1$; leaving out one element, get two sets of size 2^{k-1} which are "nice," but this is not enough – now use the elements that have not yet been used to get a third nice set of size 2^{k-1}!).

Extra Credit Problems.

A. Let $a_1, a_2, \ldots, a_{100}$ be a sequence of length 100 in $\mathbb{N}$. Show that there is a non-trivial subsequence of this sequence whose sum is divisible by 100. In other words, show that there exists an integer $N \geq 1$ and integers $1 \leq i_1 < i_2 < \cdots < i_N \leq 100$ such that $a_{i_1} + a_{i_2} + \cdots + a_{i_n}$ is divisible by 100.

Hint: Use the pigeon-whole principle as applied to the remainders of the numbers when divided by 100.

B. It is a fact, due to Chebyshev, that for any integer $n \geq 1$, there exists a prime in the interval $(n, 2n]$. Use this fact to prove that the *harmonic numbers* defined by

$$H_k = \sum_{j=1}^{k} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k},$$

are not integers for $k > 1$.

C. Recalling the Fibonacci numbers from the previous homework, show that

$$F_n = F_k F_{n-k} + F_{k-1} F_{n-k-1} \qquad \text{for } 1 \leq k \leq n-1.$$

SuperExtra Credit Problems.

D. Let $a_1, a_2, \ldots, a_{51}$ be integers with $1 \leq a_i \leq 100$ for all $1 \leq i \leq 51$. Prove that there exists $i \neq j$ such that $a_i | a_j$.

Super Duper Extra Credit Problems.

E. Let $n \geq 1$ be a positive integer. Suppose you have $2n+1$ not necessarily distinct positive integers such that whenever one of the numbers is removed, the remaining $2n$ numbers can be divided into two groups of size n that add up to the same number. Show that the numbers are all the same.

To state this more formally, let $S = \{1, 2, 3, \ldots, 2n, 2n+1\}$. Suppose $f : S \to \mathbb{N}$ is a map such that for all $x \in S$, there exist sets $T, U \subset S \setminus \{x\}$ such that $T \cap U = \emptyset$, $|T| = |U| = n$, and $\sum_{t \in T} f(t) = \sum_{u \in U} f(u)$. Show that f is a constant function i.e. for all $s_1, s_2 \in S$, $f(s_1) = f(s_2)$.

Hint: It is relatively easy to prove that all the numbers have the same parity. Is this helpful at all?