UMASS AMHERST MATH 300 FALL '05, F. HAJIR

HOMEWORK 4: EQUIVALENCE RELATIONS AND PARTITIONS

HW 4 is due in class on Thursday October 20.

1. Reading

You should read Part 5 of Farshid's notes.

2. Problems from Gilbert/Vanstone

3. Problems from Farshid's Brain

Part 1. **(Equivalence) Relations and Partitions**

1. Consider the following relation on $\mathbb{Z}$: if $a, b \in \mathbb{Z}$, then $a \sim b$ if and only if $a \cdot b$ is even. Prove or Disprove: $\sim$ defines an equivalence relation on $\mathbb{Z}$.

2. Suppose X is a set and $\sim$ is an equivalence relation on X. Suppose $x, z \in X$. Prove that either $\mathrm{Eq}(x) = \mathrm{Eq}(z)$ or else $\mathrm{Eq}(x) \cap \mathrm{Eq}(z) = \emptyset$. [Hint: In other words, show that $\mathrm{cl}(x) \neq \mathrm{cl}(z) \Rightarrow \mathrm{cl}(x) \cap \mathrm{cl}(z) = \emptyset$.

3. Suppose $\sim$ is an equivalence relation on a set X with graph R. For $x \in X$, show that $R_{x,\bullet} = R_{\bullet,x}$.

4. If X is a set and $\sim$ is an equivalence relation on it, then we have a map $X \to X/\sim$ defined by $x \mapsto \mathrm{cl}(x)$. Show that this map is surjective. (Hint: this is a very easy problem; it requires only a careful examination of the definitions involved).

5. Show that every quotient of a countable set is countable, i.e. show that if S is a countable set and $\sim$ is an equivalence relation on S, then $\widetilde{S}$ is also countable.

6. Suppose $\Delta \subseteq \mathcal{P}(X) \setminus \emptyset$ is a collection of non-empty subsets of X. Show that Δ is a partition of X if and only if for every $x \in X$ there exists a unique $S \in \Delta$ such that $x \in S$.

7. Suppose $X = \mathbb{Z}$ is the set of integers, and let n be a positive integer. Define an equivalence relation on X as follows. For $a, b \in \mathbb{Z}$, we write $a \sim_n b$ if and only if n divides $a - b$, i.e. if and only if $a - b = nk$ for some $k \in \mathbb{Z}$. This equivalence relation is called congruence modulo n. The more common notation is $a \equiv b \bmod n$.
(i) Show that this really is an equivalence relation.
(ii) Show that if $n = 1$, then all integers are equivalent to each other.
(iii) Show that if $n = 2$, then the resulting equivalence relation is the parity equivalence relation discussed above.
(iv) Show that under $\sim_n$, $\mathbb{Z}$ breaks up into n equivalence relations corresponding to the n possible remainders $0, 1, 2, \cdots, n-1$ for division by n. Thus, the set $\mathrm{Rem}(n) =$

$\{0, 1, 2, \ldots, n-1\}$ is a natural indexing set for the partition of $\mathbb{Z}$ corresponding to congruence modulo n.

(v) Use (iv) to show that we can write $\mathbb{Z}/\sim_n = \{X_0, X_1, \cdots, X_{n-1}\}$ where

$$X_j = \{a \in \mathbb{Z} | \text{ the remainder of } a \text{ divided by } n \text{ is } j\}.$$

8. In this problem, you will show that the two concepts of "equivalence relation on a set X" and "partition of X" are really the same concept, i.e. you will prove the fundamental theorem of equivalence relations.

(a) Suppose X is a set equipped with an equivalence relation $\sim$. Show the the set of equivalence classes of X under $\sim$ is a partition of X.

(b) Conversely, suppose $\Delta = \{X_\alpha | \alpha \in A\}$ is a partition of a set X. Now define a relation $\sim_\Delta$ on X as follows: if $x, y \in X$, then $x \sim_\Delta y$ if and only if there exists $\alpha \in A$ such that $x, y \in X_\alpha$. Prove that $\sim_\Delta$ is an equivalence relation on X.

(c) Prove that if $\sim$ is an equivalence relation on X, then $\sim_{\widetilde{X}} = \sim$.

(d) Prove that if Δ is a partition of X, then $X/\sim_\Delta$ is just Δ.

9. With the fundamental theorem of equivalence relations we established that equivalence relations on X and partitions on X are basically the same object and give rise to a map $X \to \widetilde{X}$. In this problem, you will how a map $X \to Y$ induces an equivalence relation on X.

Suppose X, Y are sets and $f : X \to Y$ is an arbitrary map. For $y \in Y$, the fiber of f at y (or above y) is defined to be the set $\mathbf{f}^{-1}(y) = \{x \in X \mid f(x) = y\}$. The notation $\mathbf{f}^{-1}(y)$ should not be confused with the inverse function f^{-1}. Note that we are not assuming that f is bijective. Thus, the set $\mathbf{f}^{-1}(y)$ could be empty or it could have more than one element. **If** f **is** bijective, however, **then** for each $y \in Y$, $\mathbf{f}^{-1}(y)$ is a singleton set whose only element is $f^{-1}(z)$.

Let Δ be the set of **non-empty** fibers of f, i.e. $\Delta = \{\mathbf{f}^{-1}(y) \mid y \in \text{Image}(f)\}$.

(a) Show that Δ is a partition of X.

(b) Define a relation $\sim$ on X by the rule $x \sim x'$ if and only if $f(x) = f(x')$ for $x, x' \in X$. Prove that $\sim$ defines an equivalence relation on X.

(c) For the relation $\sim$ defined in (b), prove that the equivalence classes of $\sim$ coincide with the elements of Δ, in other words, the non-empty fibers of f are precisely the equivalence classes of the equivalence relation $\sim$.

(d) For the equivalence relation $\sim$ defined in (b), define a simple and natural bijection $\varphi : \text{Image}(f) \to \widetilde{X}$ explicitly. Also define explicity the inverse map $\varphi^{-1} : \widetilde{X} \to \text{Image}(f)$ making sure to show that this map is well-defined.

(e) Suppose $n \geq 1$ is a positive integer, and recall that $\text{Rem}(n) = \{0, 1, 2, \ldots, n-1\}$. Let $X = \mathbb{Z}$ and $Y = \text{Rem}(n)$. Define the *reduction map modulo* n by $f : \mathbb{Z} \to \text{Rem}(n)$ where $f(x)$ is the remainder when x is divided by n, i.e. $f(x) = r$ where $x = nq + r$ for some $q \in \mathbb{Z}$ and $0 \leq r \leq n-1$. Show that the fibers of the map f are precisely the equivalence classes of congruence modulo n, and thus the equivalence relation one obtains on $\mathbb{Z}$ by the method of (b) is just congruence modulo n.

(f) Suppose $X = C^\infty(\mathbb{R})$ is the set consisting of all infinitely-differentiable functions on $\mathbb{R}$, i.e. functions $g : \mathbb{R} \to \mathbb{R}$ such that $g', g'', \ldots, g^{(n)}$ are well-defined functions from $\mathbb{R}$ to $\mathbb{R}$ for all $n \geq 1$. Define a relation $\sim$ on X as follows: for $g, h \in X$, $g \sim h$ if and only if $g - h$ is a constant function i.e. if and only if there exists $c \in R$ such that $g(x) - h(x) = c$ for all

$x \in \mathbb{R}$. There is a very familiar map $D : X \Rightarrow X$ such that the fibers of D are exactly the $\sim$-equivalence classes of $\sim$. What is D?! Explain. Letting $z : \mathbb{R} \to \mathbb{R}$ be the zero map, i.e. $z(x) = 0$ for all $x \in \mathbb{R}$, what is the fiber $\mathbf{D}^{-1}(z)$ above z?